Convergence of Common Fixed Point for Asymptotically Quasi-Nonexpansive Mappings in Convex Metric Spaces

Gurucharan Singh Saluja and Hemant Kumar Nashine

Abstract. In this paper, the necessary and sufficient conditions for three-step iterative sequences with errors to converge to a common fixed point for three asymptotically quasi-nonexpansive mappings is established in convex metric spaces. The results of this paper are generalizations and improvements of the corresponding results of Chang [1] - [3], Kim et al. [8], Liu [9] - [11], Ghosh and Debnath [4], Xu and Noor [15], Shahzad and Udomene [13], Khan and Takahashi [6] and Khan and Ud-din [7].

1. Introduction and Preliminaries

Throughout this paper, we assume that $E$ is a metric space, $F(T)$ and $D(T)$ are the set of fixed points and domain of $T$ respectively and $\mathbb{N}$ is the set of all positive integers.

Definition 1.1 ([8]). Let $T : D(T) \subset E \rightarrow E$ be a mapping.

(1) The mapping $T$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in D(T).$$

(2) The mapping $T$ is said to be quasi-nonexpansive if

$$d(Tx, p) \leq d(x, p), \quad \forall x \in D(T), \forall p \in F(T).$$

(3) The mapping $T$ is said to be asymptotically nonexpansive if there exists a sequence $r_n \in [0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ such that

$$d(T^n x, T^n y) \leq (1 + r_n)d(x, y), \quad \forall x, y \in D(T), \forall n \in \mathbb{N}.$$

(4) The mapping $T$ is said to be asymptotically quasi-nonexpansive if there exists a sequence $r_n \in [0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ such that

$$d(T^n x, p) \leq (1 + r_n)d(x, p), \quad \forall x \in D(T), \forall p \in F(T), \forall n \in \mathbb{N}.$$
Remark 1.1. From the definition 1.1, it follows that if $F(T)$ is nonempty, then a nonexpansive mapping is quasi-nonexpansive, and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. But the converse does not hold.

The iterative approximation problems of fixed points for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings in Hilbert spaces or Banach spaces have been studied extensively by many others. In 1973, Petryshyn and Williamson [12] obtained a necessary and sufficient condition for Picard iterative sequences and Mann iterative sequences to converge to a fixed point for quasi-nonexpansive mappings and later, the result of [12] was extended by Ghosh and Debnath [4] to Ishikawa iterative sequences. Recently, Chang [1] - [3] has proved some other kinds of necessary and sufficient conditions for Ishikawa iterative sequences with errors to converge to a fixed point for asymptotically nonexpansive mappings and Xu and Noor [15] have established a convergence theorem of three-step iterative sequences with errors for asymptotically nonexpansive mappings in uniformly convex Banach spaces. In particular, Liu [9] obtained a necessary and sufficient condition for Ishikawa iterative sequences of asymptotically quasi-nonexpansive mappings in Banach spaces to converge to a fixed point and he [10] has also extended his result [9] to Ishikawa iterative sequences with errors. Furthermore, Kim et al. [8] extended the result of Liu [9] to modified three-step iterative sequences with mixed errors.


The purpose of this paper is to study some necessary and sufficient conditions for three-step iterative sequences with errors to converge to common fixed points for three asymptotically quasi-nonexpansive mappings in convex metric spaces. The results of this paper are generalization and improvements of the corresponding results in Chang [1] - [3], Ghosh and Debnath [4], Ud-din and Khan [5], Khan and Takahashi [6], Khan and Ud-din [7], Kim et al. [8], Liu [9] - [11], Shahzad and Udomene [13] and Xu and Noor [15].

For the sake of convenience, we first recall some definitions and notations.

Definition 1.2. Let $(E,d)$ be a metric space and $I = [0, 1]$. A mapping $W : E^3 \times I^3 \to E$ is said to be a convex structure on $E$ if it satisfies the following conditions: for all $u, x, y, z \in E$ and for all $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$, 

\[ W(u, x, y, z) = \alpha u + \beta x + \gamma y. \]
(1) \( W(x, y, z; \alpha, 0, 0) = x, \)

(2) \( d(u, W(x, y, z; \alpha, \beta, \gamma)) \leq \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z). \)

If \((E, d)\) is a metric space with a convex structure \( W \), then \((E, d, W)\) is called a convex metric space and denotes it by \((E, d, W)\).

**Remark 1.2.** Every linear normed space is a convex metric space, where a convex structure \( W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z \), for all \( x, y, z \in E \) and \( \alpha, \beta, \gamma \in I \) with \( \alpha + \beta + \gamma = 1 \). But there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Takahashi [14]).

**Definition 1.3.** (1) Let \((E, d, W)\) be a convex metric space, \( T_1, T_2, T_3 : E \to E \) be mappings and let \( x_1 \in E \) be a given point. Then the sequence \( \{x_n\} \) defined by

\[
\begin{align*}
    x_{n+1} &= W(x_n, T_n^n y_n, u_n; a_n, b_n, c_n), \\
y_n &= W(x_n, T_n^n z_n, v_n; a'_n, b'_n, c'_n), \\
z_n &= W(x_n, T_n^n w_n; a''_n, b''_n, c''_n),
\end{align*}
\]

is called the three-step iterative sequence with errors for three mappings \( T_1, T_2, T_3 \), where \( \{a_n\}, \{a'_n\}, \{a''_n\}, \{b_n\}, \{b'_n\}, \{b''_n\}, \{c_n\}, \{c'_n\} \) and \( \{c''_n\} \) are nine sequences in \([0,1]\) satisfying the following conditions:

\[
a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1, \quad \forall n \in \mathbb{N},
\]

and \( \{u_n\}, \{v_n\}, \{w_n\} \) are three bounded sequences in \( E \).

(2) In (1.1), if \( b''_n = c''_n = 0 \), for all \( n = 1, 2, \ldots \), then \( z_n = x_n \) and \( T_1 = T_2 = T_3 = T \). Then the sequence \( \{x_n\} \) defined by

\[
\begin{align*}
    x_{n+1} &= W(x_n, T_n^n y_n, u_n; a_n, b_n, c_n), \\
y_n &= W(x_n, T_n^n x_n, v_n; a'_n, b'_n, c'_n),
\end{align*}
\]

is called the Ishikawa type (or two-step) iterative sequence with errors for the mapping \( T \), where \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\} \) and \( \{c'_n\} \) are six sequences in \([0,1]\) satisfying the conditions:

\[
a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \quad \forall n \in \mathbb{N},
\]

and \( \{u_n\}, \{v_n\} \) are two bounded sequences in \( E \).

2. **Main Results**

In order to prove our main result, we will first prove the following lemma.

**Lemma 2.1.** Let \((E, d, W)\) be a convex metric space, \( T_1, T_2, T_3 : E \to E \) be three asymptotically quasi-nonexpansive mappings satisfying \( \sum_{n=1}^{\infty} r_n < \infty \) where \( \{r_n\} \) is the sequence appeared in Definition 1.1, and \( F = \bigcap_{i=1}^{3} F(T_i) \) be a nonempty set. For a given \( x_1 \in E \), let \( \{x_n\} \) be the three-step iterative sequences with errors defined by (1.1). Then
(a) \(d(x_{n+1}, p) \leq (1 + r_n)^3d(x_n, p) + B_n, \forall p \in F, \ n \in \mathbb{N},\)
where \(B_n = A_nb_n(1 + r_n) + c_n d(u_n, p), \ A_n = b'_n c'_n(1 + r_n)d(w_n, p) + c'_nd(v_n, p)\)
and \(\{u_n\}, \{v_n\}, \{w_n\}\) are three bounded sequences in \(E.\)
(b) there exists a constant \(M > 0\) such that
\(d(x_m, p) \leq M.d(x_n, p) + M.\sum_{j=n}^{m-1} B_j, \forall p \in F, m > n.\)

**Proof.** (a) Let \(p \in F = \bigcap_{i=1}^3 F(T_i).\) Since \(T_i(i = 1, 2, 3)\) is asymptotically quasi-nonexpansive, we have

\[
d(x_{n+1}, p) = d(W(x_n, T^m_1 y_n, u_n; a_n, b_n, c_n), p) \\
\leq a_n d(x_n, p) + b_n d(T^m_1 y_n, p) + c_n d(u_n, p) \\
\leq a_n d(x_n, p) + b_n(1 + r_n)d(y_n, p) + c_n d(u_n, p)
\]

\[
d(y_n, p) = d(W(x_n, T^m_2 z_n, v_n; a'_n, b'_n, c'_n), p) \\
\leq a'_n d(x_n, p) + b'_n d(T^m_2 z_n, p) + c'_n d(v_n, p) \\
\leq a'_n d(x_n, p) + b'_n(1 + r_n)d(z_n, p) + c'_n d(v_n, p)
\]

and

\[
d(z_n, p) = d(W(x_n, T^m_3 x_n, w_n; a''_n, b''_n, c''_n), p) \\
\leq a''_n d(x_n, p) + b''_n d(T^m_3 x_n, p) + c''_n d(w_n, p) \\
\leq a''_n d(x_n, p) + b''_n(1 + r_n)d(x_n, p) + c''_n d(w_n, p) \\
\leq a''_n(1 + r_n)d(x_n, p) + b''_n(1 + r_n)d(x_n, p) + c''_n d(w_n, p) \\
\leq (a''_n + b''_n)(1 + r_n)d(x_n, p) + c''_n d(w_n, p) \\
= (1 - c''_n)(1 + r_n)d(x_n, p) + c''_n d(w_n, p) \\
\leq (1 + r_n)d(x_n, p) + c''_n d(w_n, p)
\]
Substituting (2.3) into (2.2), we have

\begin{equation}
\begin{aligned}
d(y_n, p) & \leq a'_n d(x_n, p) + b'_n (1 + r_n)[(1 + r_n) d(x_n, p) + c''_n d(w_n, p)] \\
& \quad + c'_n d(v_n, p) \\
& \leq a'_n d(x_n, p) + b'_n (1 + r_n)^2 d(x_n, p) + b'_n (1 + r_n) c''_n d(w_n, p) \\
& \quad + c'_n d(v_n, p) \\
& \leq a'_n d(x_n, p) + b'_n (1 + r_n)^2 d(x_n, p) + b'_n (1 + r_n) c''_n d(w_n, p) \\
& \quad + c'_n d(v_n, p) \\
& \leq a'_n (1 + r_n)^2 d(x_n, p) + b'_n (1 + r_n)^2 d(x_n, p) + b'_n (1 + r_n) c''_n d(w_n, p) \\
& \quad + c'_n d(v_n, p) \\
& \leq (a'_n + b'_n) (1 + r_n)^2 d(x_n, p) + b'_n (1 + r_n) c''_n d(w_n, p) + c'_n d(v_n, p) \\
& = (1 - c'_n) (1 + r_n)^2 d(x_n, p) + b'_n (1 + r_n) c''_n d(w_n, p) + c'_n d(v_n, p) \\
& \leq (1 + r_n)^2 d(x_n, p) + A_n
\end{aligned}
\end{equation}

where \( A_n = b'_n (1 + r_n) c''_n d(w_n, p) + c'_n d(v_n, p) \). And again, substituting (2.4) into (2.1), it follows that

\begin{equation}
\begin{aligned}
d(x_{n+1}, p) & \leq a_n d(x_n, p) + b_n (1 + r_n) [(1 + r_n)^2 d(x_n, p) + A_n] \\
& \quad + c_n d(u_n, p) \\
& \leq a_n d(x_n, p) + b_n (1 + r_n)^3 d(x_n, p) + b_n (1 + r_n) A_n + c_n d(u_n, p) \\
& \leq a_n (1 + r_n)^3 d(x_n, p) + b_n (1 + r_n)^3 d(x_n, p) + b_n (1 + r_n) A_n + c_n d(u_n, p) \\
& \leq (a_n + b_n) (1 + r_n)^3 d(x_n, p) + b_n (1 + r_n) A_n + c_n d(u_n, p) \\
& = (1 - c_n) (1 + r_n)^3 d(x_n, p) + b_n (1 + r_n) A_n + c_n d(u_n, p) \\
& \leq (1 + r_n)^3 d(x_n, p) + B_n
\end{aligned}
\end{equation}

where \( B_n = b_n (1 + r_n) A_n + c_n d(u_n, p) \). This completes the proof of (a).
(b) If $x \geq 0$, then $1 + x \leq e^x$ and $(1 + x)^3 \leq e^{3x}$. Therefore from (a) we can obtain that
\[
d(x_m, p) \leq (1 + r_{m-1})^3 d(x_{m-1}, p) + B_{m-1}
\leq e^{3r_{m-1}} d(x_{m-1}, p) + B_{m-1}
\leq e^{3r_{m-1}} [e^{3r_{m-2}} d(x_{m-2}, p) + B_{m-2}] + B_{m-1}
\leq e^{3(r_{m-1} + r_{m-2})} d(x_{m-2}, p) + e^{3r_{m-1}} B_{m-2} + B_{m-1}
\leq e^{3(r_{m-1} + r_{m-2})} d(x_{m-2}, p) + e^{3r_{m-1}} [B_{m-1} + B_{m-2}]
\leq \ldots \ldots
\leq \ldots \ldots
\leq e^{3(r_{m-1} + r_{m-2} + \cdots + r_n)} d(x_n, p)
+ e^{3(r_{m-1} + r_{m-2} + \cdots + r_n)} [B_{m-1} + B_{m-2} + \cdots + B_n]
\leq e^{3 \sum_{j=n}^{m-1} r_j d(x_n, p)} + e^{3 \sum_{j=n}^{m-1} r_j} \sum_{j=n}^{m-1} B_j
\leq M.d(x_n, p) + M. \sum_{j=n}^{m-1} B_j,
\]
where $M = e^{3 \sum_{j=n}^{m-1} r_j}$. This completes the proof of (b).

Lemma 2.1 [10]. Let the number of sequences $\{a_n\}$, $\{b_n\}$ and $\{\lambda_n\}$ satisfy that $a_n \geq 0$, $b_n \geq 0$, $\lambda_n \geq 0$, $a_{n+1} \leq (1 + \lambda_n) a_n + b_n$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then

(a) $\lim_{n \to \infty} a_n$ exists.
(b) If $\lim \inf_{n \to \infty} a_n = 0$, then $\lim_{n \to \infty} a_n = 0$.

Now, we are in a position to prove the main results. $D_d(y, S)$ denotes the distance from $y$ to set $S$, that is, $D_d(y, S) = \inf \{d(y, s) : s \in S\}$.

Theorem 2.1. Let $(E, d, W)$ be a complete convex metric space, $T_1, T_2, T_3 : E \to E$ be three asymptotically quasi-nonexpansive mappings and $F = \bigcap_{i=1}^{3} F(T_i)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the three-step iterative sequence with errors defined by (1.1) and $\{r_n\}$, $\{c_n\}$, $\{c'_n\}$, $\{c''_n\}$ be four sequences satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} r_n < \infty$,
(ii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$, $\sum_{n=1}^{\infty} c''_n < \infty$,

where $\{r_n\}$ is a sequence appeared in Definition 1.1 and $\{c_n\}$, $\{c'_n\}$, $\{c''_n\}$ are three sequences appeared in (1.1). Then the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, 3\}$ if and only if
\[
\lim_{n \to \infty} \inf D_d(x_n, F) = 0.
\]

Proof. The necessity is obvious. Now, we prove the sufficiency. Suppose that the condition $\lim \inf_{n \to \infty} D_d(x_n, F) = 0$ is satisfied. Then from Lemma
2.1(a), we have

\[(2.5) \quad d(x_{n+1}, p) \leq (1 + r_n)^3 d(x_n, p) + B_n, \quad \forall p \in F, \quad \forall n \in \mathbb{N},\]

where \( B_n = b_n(1 + r_n)A_n + c_n d(u_n, p) \) and \( A_n = b'_n(1 + r_n)c'_n d(w_n, p) + c''_n d(v_n, p) \). Since \( 0 \leq b_n, b'_n \leq 1 \), \( \sum_{n=1}^{\infty} r_n < \infty \), \( \sum_{n=1}^{\infty} c_n < \infty \), \( \sum_{n=1}^{\infty} c'_n < \infty \), \( \sum_{n=1}^{\infty} c''_n < \infty \) and \( \{u_n\}, \{v_n\}, \{w_n\} \) are three bounded sequences, we have \( \sum_{n=1}^{\infty} A_n < \infty \) and so \( \sum_{n=1}^{\infty} B_n < \infty \). From (2.5) we can obtain that

\[D_d(x_{n+1}, F) \leq (1 + r_n)^3 D_d(x_n, F) + B_n.\]

Since \( \liminf_{n \to \infty} D_d(x_n, F) = 0 \), by Lemma 2.2, we have

\[
\lim_{n \to \infty} D_d(x_n, F) = 0.
\]

Now, we will prove that \( \{x_n\} \) is a Cauchy sequence. Let \( \varepsilon > 0 \). By Lemma 2.1(b), there exists a constant \( M > 0 \) such that

\[(2.6) \quad d(x_m, p) \leq M \cdot d(x_n, p) + M \sum_{j=n}^{m-1} B_j, \quad \forall p \in F, \quad m > n.
\]

Since \( \lim_{n \to \infty} D_d(x_n, F) = 0 \) and \( \sum_{n=1}^{\infty} B_n < \infty \), there exists a constant \( N_1 \) such that for all \( n \geq N_1 \),

\[D_d(x_n, F) < \frac{\varepsilon}{4M} \text{ and } \sum_{j=N_1}^{\infty} B_j < \frac{\varepsilon}{6M}.\]

We note that there exists \( p_1 \in F \) such that \( d(x_{N_1}, p_1) < \frac{\varepsilon}{3M} \). It follows that from (2.6) that for all \( m > n > N_1 \),

\[(2.7) \quad d(x_m, x_n) \leq d(x_m, p_1) + d(x_n, p_1)
\]

\[\leq M \cdot d(x_{N_1}, p_1) + M \sum_{j=N_1}^{m-1} B_j + Md(x_{N_1}, p_1) + M \sum_{j=N_1}^{n-1} B_j
\]

\[< M \cdot \frac{\varepsilon}{3M} + M \cdot \frac{\varepsilon}{6M} + M \cdot \frac{\varepsilon}{3M} + M \cdot \frac{\varepsilon}{6M}
\]

\[= \varepsilon.
\]

Since \( \varepsilon \) is an arbitrary positive number, (2.7) implies that \( \{x_n\} \) is a Cauchy sequence. From the completeness of this, \( \lim_{n \to \infty} x_n \) exists. Let \( \lim_{n \to \infty} x_n = p \). It will be proven that \( p \) is a common fixed point. Let \( \bar{\varepsilon} > 0 \). Since \( \lim_{n \to \infty} x_n = p \), there exists a natural number \( N_2 \) such that for all \( n \geq N_2 \),

\[(2.8) \quad d(x_n, p) < \frac{\bar{\varepsilon}}{2(2 + r_1)}.
\]

\( \lim_{n \to \infty} D_d(x_n, F) = 0 \) implies that there exists a natural number \( N_3 \geq N_2 \) such that for all \( n \geq N_3 \),

\[(2.9) \quad D_d(x_n, F) < \frac{\bar{\varepsilon}}{3(4 + 3r_1)}.
\]
Therefore, there exists a $p^* \in F$ such that
\begin{equation}
(2.10) \quad d(x_{N_3}, p^*) < \frac{\bar{\epsilon}}{2(4 + 3r_1)}.
\end{equation}

From (2.8) and (2.10), we have for any $i \in I$
\begin{align*}
d(T_ip, p) &\leq d(T_ip, p^*) + d(p^*, T_ix_{N_3}) + d(T_ix_{N_3}, p^*) + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\
&= d(T_ip, p^*) + 2d(T_ix_{N_3}, p^*) + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\
&\leq (1 + r_1)d(p, p^*) + 2(1 + r_1)d(x_{N_3}, p^*) + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\
&\leq (1 + r_1)[d(p, x_{N_3}) + d(x_{N_3}, p^*)] + 2(1 + r_1)d(x_{N_3}, p^*) \\
&\quad + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\
&= (2 + r_1)d(x_{N_3}, p) + (4 + 3r_1)d(x_{N_3}, p^*) \\
&< (2 + r_1) \frac{\bar{\epsilon}}{2(2 + r_1)} + (4 + 3r_1) \frac{\bar{\epsilon}}{2(4 + 3r_1)} \\
&= \bar{\epsilon}.
\end{align*}

Since $\bar{\epsilon}$ is an arbitrary positive number, this implies that $T_ip = p$. Hence $p \in F(T_i)$ for all $i \in I$ and so $p \in F = \bigcap_{i=1}^{3} F(T_i)$. Thus the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, 3\}$. This completes the proof. \hfill \Box

In (1.1), if $T_1 = T_2 = T_3 = T$, $b_n' = c_n'' = 0$ for all $n = 1, 2, \ldots$, then $z_n = x_n$. Therefore, the following corollary can be obtained from Theorem 2.1 immediately.

**Corollary 2.1.** Let $(E, d, W)$ be a complete convex metric space, $T : E \to E$ be an asymptotically quasi-nonexpansive mapping and $F(T)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the Ishikawa type iterative sequence with errors defined by (1.2) and $\{r_n\}, \{c_n\}, \{c_n'\}$ be three sequences satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} r_n < \infty$,
(ii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c_n' < \infty$,

where $\{r_n\}$ is a sequence appeared in Definition 1.1 and $\{c_n\}, \{c_n'\}$ are two sequences appeared in (1.2). Then the iterative sequence $\{x_n\}$ converges to a fixed point of $T$ if and only if
\[\liminf_{n \to \infty} D_d(x_n, F(T)) = 0.\]

By using the same method in Theorem 2.1, we can easily obtain the following theorem.

**Theorem 2.2.** Let $(E, d, W)$ be a complete convex metric space, $T_1, T_2, T_3 : E \to E$ be three quasi-nonexpansive mappings and $F = \bigcap_{i=1}^{3} F(T_i)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the three-step iterative
sequence with errors defined by:

\[ x_{n+1} = W(x_n, T_1 y_n, u_n; a_n, b_n, c_n), \]

\[ y_n = W(x_n, T_2 z_n, v_n; a'_n, b'_n, c'_n), \]

\[ z_n = W(x_n, T_3 x_n, w_n; a''_n, b''_n, c''_n), \quad \forall n \in \mathbb{N}, \]

and \( \{c_n\}, \{c'_n\}, \{c''_n\} \) are three sequences satisfying the following condition:

(i) \( \sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c'_n < \infty, \sum_{n=1}^{\infty} c''_n < \infty, \)

where \( \{c_n\}, \{c'_n\}, \{c''_n\} \) are three sequences appeared in (1.3). Then the iterative sequence \( \{x_n\} \) converges to a common fixed point of \( \{T_i : i = 1, 2, 3\} \) if and only if

\[ \liminf_{n \to \infty} D_d(x_n, F) = 0. \]

From Theorem 2.1, we can also obtain the following result for the Banach space.

**Theorem 2.3.** Let \( E \) be a real Banach space, \( T_1, T_2, T_3 : E \to E \) be three asymptotically quasi-nonexpansive mappings satisfying the condition (i) in Theorem 2.1 and \( F = \bigcap_{i=1}^{3} F(T_i) \) be a nonempty set. Let \( \{x_n\} \) be the three-step iterative sequence with errors defined by

\[ x_1 \in E, \]

\[ x_{n+1} = a_n x_n + b_n T_1^n y_n + c_n u_n, \]

\[ y_n = a'_n x_n + b'_n T_2^n z_n + c'_n v_n, \]

\[ z_n = a''_n x_n + b''_n T_3^n x_n + c''_n w_n, \quad \forall n \in \mathbb{N}, \]

where \( \{u_n\}, \{v_n\}, \{w_n\} \) are three bounded sequences in \( E \) and \( \{a_n\}, \{a'_n\}, \{a''_n\}, \{b_n\}, \{b'_n\}, \{b''_n\}, \{c_n\}, \{c'_n\} \) and \( \{c''_n\} \) are nine sequences in \([0,1]\) satisfying \( a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1, \forall n \in \mathbb{N}, \) and \( \sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c'_n < \infty, \sum_{n=1}^{\infty} c''_n < \infty. \) Then the iterative sequence \( \{x_n\} \) converges to a common fixed point of \( \{T_i : i = 1, 2, 3\} \) if and only if

\[ \liminf_{n \to \infty} D(x_n, F) = 0. \]

where \( D_d(y, S) = \inf\{d(y, s) : s \in S\}. \)

**Proof.** Since \( E \) is a Banach space, it is a complete convex metric space with a convex structure \( W(x, y, z : \alpha, \beta, \gamma) := \alpha x + \beta y + \gamma z, \) for all \( x, y, z \in E \) and for all \( \alpha, \beta, \gamma \in [0,1] \) with \( \alpha + \beta + \gamma = 1. \) Therefore, the conclusion of Theorem 2.3 can be obtained from Theorem 2.1 immediately. \( \square \)

**Remark 2.1.** (1) Theorem 2.1 and 2.2 are two new convergence theorems of three-step iterative sequences with errors for nonlinear mappings in convex metric spaces. These two theorems generalize and improves the corresponding results of [9]- [11], [1]- [3] and [4, 6, 7, 12, 13, 15].
Theorem 2.3 generalizes and improves the corresponding results of Kim et al. [8], Liu [10, 11], Shahzad and Udomene [13], Khan and Takahashi [6], Khan and Ud-din [7] and Xu and Noor [15].

REFERENCES


Gurucharan Singh Saluja
Department of Mathematics & Information Technology
Govt. College of Science
Raipur-492101 (Chhattisgarh)
India
E-mail address: saluja_1963@rediffmail.com

Hemant Kumar Nashine
Department of Mathematics
Disha Institute of Management and Technology
Satya Vihar, Vidhansabha – Chandrakhuri Marg
(Baloda Bazar Road), Mandir Hasaud,
Raipur-492101 (Chhattisgarh)
India
E-mail address: hnashine@rediffmail.com
hemantnashine@rediffmail.com