The Matrix Transformations on Double Sequence Space of $\chi_\pi^2$

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Abstract. Let $\chi^2$ denote the space of all prime sense double gai sequences and $\Lambda^2$ the space of all prime sense double analytic sequences. First we show that the set $E = \{s^{(mn)} : m, n = 1, 2, 3, \ldots\}$ is a determining set for $\chi^2_\pi$. The set of all finite matrices transforming $\chi^2_\pi$ into FK-space $Y$ denoted by $\chi^2_\pi : Y$. We characterize the classes $(\chi^2_\pi : Y)$ when $Y = c_0^2, c^2, \chi^2, \ell^2, \Lambda^2$.

<table>
<thead>
<tr>
<th>$\chi^2_\pi$</th>
<th>$c_0^2$</th>
<th>$c^2$</th>
<th>$\chi^2$</th>
<th>$\ell^2$</th>
<th>$\Lambda^2$</th>
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Necessary and sufficient condition on the matrix are obtained.

1. Introduction

Throughout $w, \chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences respectively. We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$ the set of positive integers. Then $w^2$ is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [2]. Later on it was investigated by Hardy [3], Moricz [4], Moricz and Rhoades [5], Basarir and Solankan [1], Tripathy [6], Colak and Turkmenoglu [7], Turkmenoglu [8], and many others. We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

\[(a + b)^p \leq a^p + b^p\]

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence. $(s_{mn})$ is called convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m, n = 1, 2, 3, \ldots)$ (see [9]). A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{m,n} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences

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will be denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double gai sequence if $((m + n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by $\chi^2$. Let $\phi = \{\text{all finite sequences}\}$. Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \zeta_{ij}$ for all $m, n \in \mathbb{N}$,

$$\zeta_{mn} = \begin{pmatrix}
0, 0, \ldots, 0, 0, \ldots \\
0, 0, \ldots, 0, 0, \ldots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0, 0, \ldots, \pi_{mn}, -\pi_{mn}, \ldots \\
0, 0, \ldots, 0, 0, \ldots 
\end{pmatrix}$$

with $\pi_{mn}$ in the $(m, n)^{th}$ position, $-\pi_{mn}$ in the $(m + 1, n + 1)^{th}$ position and zero otherwise. An FK-space (or a metric space) $X$ is said to have AK property if $(\zeta_{mn})$ is a Schauder basis for $X$. Or equivalently $x^{[m,n]} \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous. If $X$ is a sequence space, we give the following definitions:

(i) $X' = \text{the continuous dual of } X$;
(ii) $X^\alpha = \{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\}$;
(iii) $X^\beta = \{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent, for each } x \in X\}$;
(iv) $X^\gamma = \{a = (a_{mn}) : m, n \geq 1 \sup_{M,N} |\sum_{m,n=1}^{M,N} a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\}$;
(v) let $X$ be an FK-space $\supset \phi$; then $X^f = \{f(\zeta_{mn}) : f \in X'\}$;
(vi) $X^\Lambda = \{a = (a_{mn}) : \sup_{m,n} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X\}$

$X^\alpha X^\beta, X^\gamma$ are called $\alpha-$ (or Köthe-Toeplitz) dual of $X$; $\beta-$ (or generalized-Köthe-Toeplitz) dual of $X$; $\gamma-$dual of $X$, $\Lambda-$dual of $X$ respectively.

2. Definitions and Preliminaries

$$\chi^2_{\pi} = \{x = (x_{mn}) : \left(\frac{x_{mn}}{\pi_{mn}}\right) \in \chi^2\};$$

$$\Lambda^2_{\pi} = \{x = (x_{mn}) : \left(\frac{x_{mn}}{\pi_{mn}}\right) \in \Lambda^2\}.$$

The space $\Lambda^2_{\pi}$ is a metric space with the metric

$$d(x, y) = \sup_{m,n} \left\{\left|\frac{x_{mn} - y_{mn}}{\pi_{mn}}\right|^{1/m+n} : m, n : 1, 2, 3, \ldots \right\}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in $\Lambda^2$.

The space $\chi^2_{\pi}$ is a metric space with the metric

$$d(x, y) = \sup_{m,n} \left\{\left((m + n)! \left|\frac{x_{mn} - y_{mn}}{\pi_{mn}}\right|\right)^{1/m+n} : m, n : 1, 2, 3, \ldots \right\}$$
for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in $\chi^2$.

Let $X$ be an BK-space. Then $D = D(X) = \{x \in \phi : \|x\| \leq 1\}$ we do not assume that $X \supset \phi$ (i.e.) $D = \phi \bigcap (\text{unit closed sphere in } X)$.

Let $X$ be an BK space. A subset $E$ of $\phi$ will be called a determining set for $X$ if $D(X)$ is the absolutely convex hull of $E$. In respect of a metric space $(X, d), D = \{x \in \phi : d(x, 0) \leq 1\}$.

Given a sequence $x = \{x_{mn}\}$ and an four dimensional infinite matrix

$$A = \left( a_{mn}^{jk} \right), m, n, j, k = 1, 2, \ldots \text{ then } A- \text{transform of } x \text{ is the sequence }$$

$$y = (y_{mn}) \text{ when } y_{mn} = \sum_{n=1}^{\infty} \alpha_{mn}x_{mn} (j, k = 1, 2, \ldots). \text{ Whenever }$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{mn}x_{mn} \text{ exists. }$$

Let $X$ and $Y$ be FK-spaces. If $y \in Y$ whenever $x \in X$, then the class of all matrices $A$ is denoted by $(X : Y)$.

**Lemma 2.1.** Let $X$ be a BK-space and $E$ is determining set for $X$. Let $Y$ be an FK-space and $A$ is an four dimensional infinite matrix. Suppose that either $X$ has AK or $A$ is row finite. Then $A \in (X : Y)$ if and only if (1) The columns of $A$ belong to $Y$ and (2) $A[E]$ is a bounded subset of $Y$.

### 3. Main Results

**Theorem 3.1.** Let $E$ be the set of all sequences in $\phi$ each of whose non-zero terms is

$$
\begin{pmatrix}
0, & 0, & \cdots & 0, & 0, & \cdots \\
0, & 0, & \cdots & 0, & 0, & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \\
0, & 0, & \cdots & \frac{\pi_{mn}}{(m+n)!}, & -\frac{\pi_{mn}}{(m+n)!}, & \cdots \\
0, & 0, & \cdots & 0, & 0, & \cdots
\end{pmatrix}
$$

with $\frac{\pi_{mn}}{(m+n)!}$, in the $(m, n)^{th}$, $-\frac{\pi_{mn}}{(m+n)!}$, in the $(m + 1, n + 1)^{th}$ position and zero otherwise. Then $E$ is determining set of $\chi^2\pi$.

**Proof.** Step 1. Recall that $\chi^2\pi$ is a metric space with the metric

$$d(x, y) = \sup_{mn} \left\{ \left( (m + n)! \left| \frac{x_{mn} - y_{mn}}{\pi_{mn}} \right| \right)^{1/m+n} : m, n : 1, 2, 3, \ldots \right\}$$

Let $A$ be the absolutely convex hull of $E$. Let $x \in A$.

Then $x = \sum_{m=1}^{i} \sum_{n=1}^{j} t_{mn} \pi_{mn} s^{(mn)}$ with

$$\sum_{m,n=1}^{i,j} |t_{mn}| \leq 1.$$

and $s^{(mn)} \in E$.

Then $d(x, 0) \leq |t_{11}| \pi_{11} d(s^{(11)}, 0) + \cdots + |t_{ij}| \pi_{ij} d(s^{(ij)}, 0)$. But $d(s^{(mn)}) = 1$ for $m, n = 1, 2, 3, \ldots, (i, j)$. Hence $d(x, 0) \leq \sum_{m,n=1}^{i,j} |t_{mn}| \leq 1$ by using
(4). Also $x \in \phi$. Hence $x \in D$. Thus

\[(5)\quad A \subset D\]

**Step 2:** Let $x \in D$

\[\Rightarrow x \in \phi \quad \text{and} \quad d(x, 0) \leq 1.\]

\[x = \begin{pmatrix}
2!x_{11}, & 3!x_{12}, & \cdots & (1 + n)!x_{1n}, & \cdots \\
3!x_{21}, & 4!x_{22}, & \cdots & (2 + n)!x_{2n}, & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(m + 1)!x_{m1}, & (m + 2)!x_{m2}, & \cdots & (m + n)!x_{mn}, & \cdots \\
0, & 0, & \cdots & 0, & \cdots
\end{pmatrix}\]

and

\[(6)\quad \sup \left(\frac{2! |x_{11}|^{1/2}}{0,}, \frac{3! |x_{12}|^{1/3}}{0,}, \cdots \left(\frac{(1 + n)! |x_{1n}|^{1/1+n}}{(m + n)! |x_{mn}|^{1/m+n}}\right)\right)\]

**Case (i):** Suppose that $2! |x_{11}| \geq \cdots \geq (m + n)! |x_{mn}|$.

Let $\xi_{mn} = \text{Sgn} ((m + n)! x_{mn}) = \frac{(m+n)! |x_{mn}|}{(m+n)! |x_{mn}|}$ for $m, n = 1, 2, \ldots, (i, j)$.

Take

\[S_{k\ell} \pi_{k\ell} = \begin{pmatrix}
\xi_{11}, & \xi_{12}, & \cdots & \xi_{1\ell}, & \cdots \\
\xi_{21}, & \xi_{22}, & \cdots & \xi_{2\ell}, & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\xi_{k1}, & \xi_{k2}, & \cdots & \xi_{k\ell}, & \cdots \\
0, & 0, & \cdots & 0, & \cdots
\end{pmatrix}\]

for $k, \ell = 1, 2, 3, \ldots, (i, j)$.

Then $\pi_{k\ell} S_{k\ell} \in E$ for $k, \ell = 1, 2, 3, \ldots, (i, j)$.

Also

\[x = (|2!x_{11} - 3!x_{12}| - |3!x_{21} - 4!x_{22}|) \pi_{11} S_{11} + \cdots\]

\[+ ((m + n)! x_{mn} - (m + n + 1)! x_{mn+1})\]

\[- (m + n + 1)! x_{m+1n} - (m + n + 2)! x_{m+1n+1}) \pi_{mn} S_{mn}\]

so that

\[t_{11} \pi_{11} S_{11} + \cdots + t_{mn} \pi_{mn} S_{mn},\]

\[t_{11} + \cdots + t_{mn} = |2!x_{11} - 3!x_{12}|\]

\[- (m + n + 1)! x_{m+1n} - (m + n + 2)! x_{m+1n+1} = |2!x_{11} - 3!x_{12}|\]

because

\[|(m + n + 1)! x_{m+1n} - (m + n + 2)! x_{m+1n+1}| = 0 \leq 1\]

by using (6).
Hence $x \in A$. Thus $D \subset A$.

**Case (ii):** Let $y$ be $x$ and let $2!|y_{11}| \geq \cdots \geq (m + n)!|y_{mn}|$. Express $y$ as a member of $A$ as in Case (i). Since $E$ is invariant under permutation of the terms of its members, so is $A$. Hence $x \in A$. Thus $D \subset A$. Therefore in both cases

$$D \subset A$$

From (5) and (7) $A = D$. Consequently $E$ is a determining set for $\chi^2_\pi$. This completes the proof. □

**Proposition 3.1.** $\chi^2_\pi$ has AK.

*Proof.* Let $x = (x_{mn}) \in \chi^2_\pi$ and take $x^{[mn]} = \sum_{i,j=1}^{m,n} x_{ij}\gamma_{ij}$ for all $m, n \in \mathbb{N}$. Hence

$$d(x, x^{[rs]}) = \sup_{mn} \left\{ \left( (m + n)! \left| \frac{x_{mn}}{\pi_{mn}} \right| \right)^{1/(m+n)} : m \geq r + 1, n \geq s + 1 \right\}$$

→ 0 as $m, n \to \infty$

Therefore, $x^{[rs]} \to x$ as $r, s \to \infty$ in $\chi^2_\pi$. Thus $\chi^2_\pi$ has AK. This completes proof. □

**Proposition 3.2.** An infinite matrix $A = \left( a_{mn}^{jk} \right)$ is in the class

$$A \in \left( \chi^2_\pi : c^2_0 \right) \iff \lim_{n,k \to \infty} \left( \pi_{mn}a_{mn}^{jk} \right) = 0$$

$$\iff \sup_{mn} \left| \pi_{m1}a_{m1}^{j1} + \cdots + \pi_{mn}a_{mn}^{jk} \right| < \infty.$$

*Proof.* In Lemma 3 take $X = \chi^2_\pi$ has AK property and take $Y = \left( c^2_0 \right)$ be an FK-space. Further more $\chi^2_\pi$ is a determining set $E$ (as in given Proposition 4). Also $A[E] = A(s^{(mn)}) = \left\{ \left( \pi_{m1}a_{m1}^{j1} + \cdots + \pi_{mn}a_{mn}^{jk} \right) \right\}$. Again by Lemma 3, $A \in \left( \chi^2_\pi : c^2_0 \right)$ if and only if:

(i) The columns of $A$ belong to $c^2_0$, and

(ii) $A(s^{(mn)})$ is a bounded subset $\chi^2_\pi$.

But the condition

(i) $\iff \left\{ \pi_{mn}a_{mn}^{jk} : j, k = 1, 2, \cdots \right\}$ is exits for all $m, n$;

(ii) $\iff \sup_{mn} \left| \pi_{m1}a_{m1}^{j1} + \cdots + \pi_{mn}a_{mn}^{jk} \right| < \infty.$

Hence we conclude that $A \in \left( \chi^2_\pi : c^2_0 \right) \iff$ conditions (8) and (9) are satisfied. □

**The following proofs are similar.** Hence we omit the proof.

**Proposition 3.3.** An infinite matrix $A = \left( a_{mn}^{jk} \right)$ is in the class

$$A \in \left( \chi^2_\pi : c^2 \right) \iff \lim_{n,k \to \infty} \left( \pi_{mn}a_{mn}^{jk} \right) \text{ exists} (m, j = 1, 2, 3, \ldots)$$
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(11) \[ \Leftrightarrow \sup_{mn} \left| \pi_{m1}a_{m1}^{j1} + \cdots + \pi_{mn}a_{mn}^{jk} \right| < \infty. \]

**Proposition 3.4.** An infinite matrix $A = (a_{mn}^{jk})$ is in the class $A \in (\chi^2_\pi : \chi^2_\pi)$ if

(12) \[ A \in (\chi^2_\pi : \chi^2_\pi) \Leftrightarrow \sup_{mn} \left( \frac{1}{\pi_{mn}(m+n)!} \left| a_{m1}^{j1} + \cdots + a_{mn}^{jk} \right| \right)^{1/(m+n)} < \infty. \]

(13) \[ \Leftrightarrow \lim_{n,k \to \infty} \left( \frac{1}{\pi_{mn}(m+n)!} \left| a_{mn}^{jk} \right| \right)^{1/(m+n)} = 0, \text{ for } m, j = 1, 2, 3, \ldots \]

(14) \[ \Leftrightarrow d\left( a_{m1}^{j1}, a_{m2}^{j2}, \ldots, a_{mn}^{jk} \right) \text{ is bounded} \]

for each metric $d$ on $\chi^2_\pi$ and for all $m, n$.

**Proposition 3.5.** An infinite matrix $A = (a_{mn}^{jk})$ is in the class $A \in (\chi^2_\pi : \ell^2)$ if

(15) \[ A \in (\chi^2_\pi : \ell^2) \Leftrightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| a_{mn}^{jk} \right| \text{ converges } (j, k = 1, 2, 3, \ldots) \]

(16) \[ \Leftrightarrow \sup_{mn} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| \pi_{mn}a_{mn}^{jk} \right| < \infty \]

**Proposition 3.6.** An infinite matrix $A = (a_{mn}^{jk})$ is in the class $A \in (\chi^2_\pi : \Lambda^2)$ if

(17) \[ A \in (\chi^2_\pi : \Lambda^2) \Leftrightarrow \sup_{mn} \left( \pi_{mn} \left| \sum_{\gamma=1}^{n} \sum_{\mu=1}^{k} a_{m\gamma}^{j\mu} \right| \right)^{1/(m+n)} < \infty \]

(18) \[ \Leftrightarrow d\left( a_{m1}^{j1}, a_{m2}^{j2}, \ldots, a_{mn}^{jk} \right) \text{ is bounded} \]

for each metric $d$ on $\Lambda^2$ and for all $m, n$.

**REFERENCES**


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