Properties of the Quasi-Conformal Curvature Tensor of Kähler-Norden Manifolds

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Abstract. The object of the present paper is to study quasi-conformally flat and parallel quasi-conformal curvature tensor of a Kähler-Norden manifold. Besides this we also study quasi-conformally semisymmetric Kähler-Norden manifolds. Finally, we mention an example to verify a Theorem of our paper.

1. Introduction

An anti-Kähler or Kähler-Norden manifold means a triple $(M^n, J, g)$ which consists of a smooth manifold $M^n$ of dimension $n = 2m$, an almost complex structure $J$ and an anti-Hermitian metric $g$ such that $\nabla J = 0$ where $\nabla$ is the Levi-Civita connection of $g$. The metric $g$ is called anti-Hermitian if it satisfies $g(JX, JY) = -g(X, Y)$ for all vector fields $X$ and $Y$ on $M^{2m}$. Then the metric $g$ has necessarily a neutral signature $(m, m)$ and $M^{2m}$ is a complex manifold and there exists a holomorphic metric on $M^{2m}$ [1]. This fact gives us some topological obstructions to an anti-Kähler manifold, for instance, all its odd Chern numbers vanish because its holomorphic metric gives us a complex isomorphism between the complex tangent bundle and its dual and a compact simply connected Kähler manifold cannot be anti-Kähler because it does not admit a holomorphic metric.

The conditions of the semisymmetry and pseudosymmetry type for the Riemann, Ricci and Weyl curvature tensors of Kählerian and para-Kählerian manifolds were studied in the papers [9, 10, 11, 12] and many others. In the present paper we extend the result of Sluka [5] in a Kähler-Norden manifold. In [4] Sluka constructed some examples of holomorphically projectively flat as well as semisymmetric and locally symmetric Kähler-Norden manifolds. The present paper is organized as follows:

After preliminaries in section 3, we study quasi-conformally flat Kähler-Norden manifolds. In section 4, we consider parallel quasi-conformal Kähler-Norden manifolds. In section 5, we study quasi-conformally semisymmetric...
Kähler-Norden manifolds. Finally, we mention an example to verify the Theorem 4.1.

2. PRELIMINARIES

By a Kählerian manifold with Norden metric (Kähler-Norden in short) [2] we mean a triple \((M, J, g)\), where \(M\) is a connected differentiable manifold of dimension \(n = 2m\), \(J\) is a \((1, 1)\)-tensor field and \(g\) is a pseudo-Riemannian metric on \(M\) satisfying the conditions

\[
J^2 = -I, \quad g(JX, JY) = -g(X, Y), \quad \nabla J = 0
\]

for every \(X, Y \in \chi(M)\) is the Lie algebra of vector fields on \(M\) and \(\nabla\) is the Levi-Civita connection of \(g\).

Let \((M, J, g)\) be a Kähler-Norden manifold. Since in dimension two such a manifold is flat, we assume in the sequel that \(\dim M \geq 4\). Let \(R(X, Y)\) be the curvature operator \([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\) and let \(\mathcal{R}\) be the Riemann-Christoffel curvature tensor, \(\mathcal{R}(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W)\). The Ricci tensor \(S\) is defined as \(S(X, Y) = \text{trace}\{Z \rightarrow \mathcal{R}(Z, X)Y\}\). These tensors have the following properties [1]

\[
\begin{align*}
\mathcal{R}(JX, JY) &= -\mathcal{R}(X, Y), \\
\mathcal{R}(JX, Y) &= \mathcal{R}(X, JY), \\
S(JY, Z) &= \text{trace}\{X \rightarrow \mathcal{R}(JX, Y)Z\}, \\
S(JX, Y) &= S(JY, X), \\
S(JX, JY) &= -S(X, Y).
\end{align*}
\]

Let \(Q\) be the Ricci operator. Then we have \(S(X, Y) = g(QX, Y)\) and

\[
QY = -\sum_i \epsilon_i \mathcal{R}(e_i, Y)e_i,
\]

where \(\{e_1, e_2, \ldots, e_n\}\) is an orthonormal basis and \(\epsilon_i\) are the indicators of \(e_i\), \(\epsilon_i = g(e_i, e_i) = \pm 1\). The notion of a quasi-conformal curvature tensor was given by Yano and Sawaki [6]. The quasi-conformal curvature tensor \(\tilde{C}\) is defined by

\[
\begin{align*}
\tilde{C}(X, Y)Z &= a\mathcal{R}(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\
&\quad - g(X, Z)QY] - r[\frac{a}{n} - \frac{1}{n-1} + 2b][g(Y, Z)X - g(X, Z)Y],
\end{align*}
\]

where \(a\) and \(b\) are constants and \(\mathcal{R}\), \(Q\) and \(r\) are Riemannain curvature tensor of type \((1, 3)\), the Ricci operator defined by \(g(QX, Y) = S(X, Y)\) and the scalar curvature, respectively. If \(a = 1\) and \(b = -\frac{1}{n-2}\), then (2) takes
the form
\[
\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2} \left[ S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \right.
\]
\[
\left. - g(X,Z)QY \right] + \frac{r}{(n-1)(n-2)} \left[ g(Y,Z)X - g(X,Z)Y \right]
\]
\[
= C(X,Y)Z,
\]
where \( C \) is the conformal curvature tensor [8]. Thus the conformal curvature tensor \( C \) is the particular case of the tensor \( \tilde{C} \). For this reason \( \tilde{C} \) is called quasi-conformal curvature tensor. A manifold \((M^n, g) \) \( (n > 3) \) shall be called quasi-conformally flat if \( \tilde{C} = 0 \). It is known [3] that a quasi conformally flat manifold is either conformally flat if \( a \neq 0 \) or Einstein if \( a = 0 \) and \( b \neq 0 \). Since they give no restrictions for manifolds if \( a = 0 \) and \( b = 0 \), it is essential for us to consider the case of \( a \neq 0 \) or \( b \neq 0 \).

Using (1) and (2) we have
\[
\sum_i \epsilon_i g(\tilde{C}(J\epsilon_i, JY)\epsilon_i, W) = b \left[ 2S(JY, JW) - r^* g(JY, W) \right]
\]
\[
- \frac{r}{n} \left[ \frac{a}{n-1} + 2b \right] g(JY, JW).
\]
This implies that
\[
\sum_i \epsilon_i \tilde{C}(J\epsilon_i, JY)\epsilon_i = b \left[ -2QY - r^* JY \right]
\]
\[
+ \frac{r}{n} \left[ -\frac{a}{n-1} + 2b \right] Y,
\]
where \( r^* \) is the *-scalar curvature, which is defined as the trace of \( JQ \). In the above we have applied the identity \( \sum_i \epsilon_i g(J\epsilon_i, \epsilon_i) = 0 \), which is a consequence of the traceless of \( J \).

The holomorphically projective curvature tensor is defined in the following way [4, 7]
\[
P(X,Y) = R(X,Y) - \frac{1}{n-2} (X \wedge_S Y - JX \wedge_S JY),
\]
where the operator \( X \wedge_S Y \) is defined by
\[
(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y, \quad Z \in \chi(M).
\]
We notice, for later use, that this tensor has the following properties
\[
\]
\[
\sum_i \epsilon_i P(e_i, Y, Z, Je_i) = 0, \quad \sum_i \epsilon_i P(X, Y, e_i, e_i) = 0,
\]
A Kähler-Norden manifold \((M, J, g)\) is holomorphically projectively flat if and only if its holomorphically projective curvature tensor \( P \) vanishes identically.
A Riemannian manifold is said to be quasi-conformally semisymmetric if 
\( \tilde{\mathcal{C}} = 0 \), where \( \mathcal{C}(X,Y) \) denotes the derivation of the tensor algebra at each point of the manifold for tangent vector fields \( X, Y \).

3. QUASI-CONFORMALLY FLAT KÄHLER-NORDEN MANIFOLDS

In this section we study quasi-conformally flat Kähler-Norden manifolds, that is, 
\( \tilde{\mathcal{C}}(X,Y)Z = 0 \). Therefore from (3) we obtain

\[
(8) \quad b[2S(JY,JW) - r^*g(JY,W)] = \frac{r}{n} \left[ \frac{a}{n-1} + 2b \right] g(JY,JW),
\]

Using (1) in (8) yields

\[
(9) \quad b[-2S(Y,W) - r^*g(JY,W)] = -\frac{r}{n} \left[ \frac{a}{n-1} + 2b \right] g(Y,W),
\]

Contracting (9) with respect to the pair of arguments \( Y, W \) (that is, taking \( Y = W = e_i \) into (9), multiplying by \( \epsilon_i \) and summing up over \( i \in \{1, \ldots, n\} \)), we have

\[
(10) \quad -2br = -\frac{r}{n} \left[ \frac{a}{n-1} + 2b \right] n.
\]

This implies

\[
(11) \quad -\frac{a}{n-1}r = 0.
\]

Since \( a \neq 0 \), then from (11) we obtain

\[
(12) \quad r = 0.
\]

Again using (12) in (9) we obtain

\[
(13) \quad S(Y,W) = -\frac{r^*}{2b}g(JY,W).
\]

Using (12), (13) in (2) we have

\[
(14) \quad \mathcal{R}(X,Y)Z = -\frac{r^*}{2a} \left[ -g(JY,Z)X + g(JX,Z)Y - g(Y,Z)JX + g(X,Z)JY \right].
\]

Also holomorphically projectively flatness implies from (5)

\[
(15) \quad \mathcal{R}(X,Y)Z = \frac{1}{n-2} \left[ S(Y,Z)X - S(X,Z)Y - S(JY,Z)JX + S(JX,Z)JY \right].
\]

Therefore from (13) and (15) it follows that

\[
(16) \quad \mathcal{R}(X,Y)Z = \frac{r^*}{2b(n-2)} \left[ -g(JY,Z)X + g(JX,Z)Y - g(Y,Z)JX + g(X,Z)JY \right].
\]
From equations (14) and (16) we obtain \( r^*[a + (n - 2)b] = 0 \). Now, \( r^*[a + (n - 2)b] = 0 \) implies either \( r^* = 0 \) or \( a + (n - 2)b = 0 \). If \( a + (n - 2)b = 0 \), then putting this into (2), we get \( \tilde{C}(X,Y)Z = aC(X,Y)Z \). So the quasi-conformally flatness and conformally flatness are equivalent in this case. Thus in view of the above result we can state the following:

**Theorem 3.1.** If a quasi-conformally flat Kähler-Norden manifold is holomorphically projectively flat, then quasi-conformally flatness and conformally flatness are equivalent provided \( r^* \neq 0 \).

**Corollary 3.1.** The Ricci tensor and curvature tensor of a quasi-conformally flat Kähler-Norden manifold \((M,J,g)\) have the shapes (13) and (14), respectively.

4. **Kähler-Norden manifolds \((M,J,g)\) with parallel quasi-conformal curvature tensor**

Assume that the quasi-conformal curvature tensor of a Kähler-Norden manifold is parallel, that is, \( \nabla \tilde{C} = 0 \). From (3) we have

\[
\sum_i \epsilon_i g(\tilde{C}(Je_i, JY)e_i, W) = b[2S(JY, JW) - r^*g(JY, W)] \\
- r \left[ \frac{a}{n} + 2b \right] g(JY, JW),
\]

where \( r^* \) is the \( * \)-scalar curvature, which is defined as the trace of \( JQ \). Taking covariant differentiation of (17) and our assumption yields

\[
0 = b[-2(\nabla Z S)(Y, W) - dr^*(Z)g(JY, W)] \\
+ \frac{dr(Z)}{n} \left[ \frac{a}{n - 1} + 2b \right] g(Y, W),
\]

since \( S(JY, JW) = -S(Y, W) \) and \( g(JY, JW) = -g(Y, W) \).

Contracting (18) with respect to the pair of arguments \( Y, W \) (that is, taking \( Y = W = e_i \) into (18), multiplying by \( \epsilon_i \) and summing up over \( i \in \{1, \ldots, n\} \)), we have

\[
-2bdr(Z) + \frac{dr(Z)}{n} \left[ \frac{a}{n - 1} + 2b \right] n = 0.
\]

Since \( a \neq 0 \), then (19) implies

\[
dr(Z) = 0.
\]

Using (20) in (18) we have

\[
(\nabla Z S)(Y, W) = -\frac{1}{2}dr^*(Z)g(JY, W).
\]

Putting \( Y = JY \) in (21) we obtain

\[
(\nabla Z S)(JY, W) = \frac{1}{2}dr^*(Z)g(Y, W).
\]
Contracting (22) with respect to the pair of arguments $Y, W$ (that is, taking $Y = W = e_i$ into (22), multiplying by $\epsilon_i$ and summing up over $i \in \{1, \ldots, n\}$), we have
\begin{equation}
\sum_i \epsilon_i \tilde{C}(Je_i, JY)e_i = b[-2QY - r^*JY] + \frac{r}{n} \left[ -\frac{a}{n-1} + 2b \right] Y,
\end{equation}
(23) $dr^*(Z) = 0$.

Again using (20) and (23) in (18) yields
\begin{equation}
(\nabla Z)S(Y, W) = 0.
\end{equation}
(24)

In view of (2), the covariant derivative $\nabla \tilde{C}$ can be expressed in the following form
\begin{equation}
(\nabla W \tilde{C})(X, Y)Z = a(\nabla W R)(X, Y)Z + b[(\nabla W S)(Y, Z)X - (\nabla W S)(X, Z)Y + g(Y, Z)(\nabla W Q)X - g(X, Z)(\nabla W Q)Y].
\end{equation}
(25)

Using (24) in (25) we obtain
\begin{equation}
(\nabla W \tilde{C})(X, Y)Z = a(\nabla W R)(X, Y)Z.
\end{equation}
(26)

Since $a \neq 0$, then in view of the above result we can state the following:

**Theorem 4.1.** A Kähler-Norden manifold $(M, J, g)$ is quasi-conformally symmetric if and only if it is locally symmetric.

5. **Quasi-conformally semisymmetric Kähler-Norden manifolds**

In this section we study Quasi-conformally semisymmetric Kähler-Norden manifolds. Assume that $\mathcal{R}.\tilde{C} = 0$. From (4) we have
\begin{equation}
\sum_i \epsilon_i \tilde{C}(Je_i, JY)e_i = b[-2QY - r^*JY] + \frac{r}{n} \left[ -\frac{a}{n-1} + 2b \right] Y,
\end{equation}
(27)

where $r^*$ is the *-scalar curvature, which is defined as the trace of $JQ$.

Since $\mathcal{R}.\tilde{C} = 0$, then from (27) we have $\mathcal{R}.Q = 0$ and hence $\mathcal{R}.S = 0$. Again
\begin{equation}
\tilde{C}(X, Y)Z = a\mathcal{R}(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
\end{equation}
(28)

where $a$ and $b$ are constants and $\mathcal{R}, Q$ and $r$ are Riemannain curvature tensor of type $(1, 3)$, the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and the scalar curvature, respectively.

By the $\mathcal{R}.\tilde{C} = 0$ and $\mathcal{R}.S = 0$ from (28) we have $\mathcal{R}.\mathcal{R} = 0$.

Conversely,
\begin{equation}
\mathcal{R}.\mathcal{R} = 0 \Rightarrow \mathcal{R}.S = 0 \Rightarrow \mathcal{R}.Q = 0 \Rightarrow \mathcal{R}.\tilde{C} = 0.
\end{equation}
(29)
From the above results we can state the following:

**Theorem 5.1.** A Kähler-Norden manifold \((M, J, g)\) is quasi-conformally semisymmetric if and only if it is semisymmetric.

In [4], Sluka proved that

**Theorem 5.2.** [4] A Kähler-Norden manifold \((M, J, g)\) is holomorphically projectively semisymmetric if and only if it is semisymmetric.

In view of Theorems 5.1 and 5.2, we can state the following:

**Theorem 5.3.** A Kähler-Norden manifold \((M, J, g)\) is quasi-conformally semisymmetric if and only if it is holomorphically projectively semisymmetric.

6. **Example**

In [4] Sluka cited an example of a Kähler-Norden manifold which is locally symmetric. This example verifies our Theorem 4.1.

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