On Variations of $m, n$-Totally Projective Abelian $p$-Groups

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Abstract. We define some new classes of $p$-torsion Abelian groups which are closely related to the definitions of $n$-totally projective, strongly $n$-totally projective and $m, n$-totally projective groups introduced by P. Keef and P. Danchev in J. Korean Math. Soc. (2013). We also study their critical properties, one of which is the so-named Nunke’s-esque property.

1. Introduction

All groups examined in the current paper will be $p$-primary Abelian, where $p$ is an arbitrary fixed prime, and $m$ and $n$ are both non-negative integers which will be used in the sequel as parameters. Most of our notions and notations will be standard being in agreement with [5] and [6]; for the specific ones, we refer the readers to [9], [10] and [11]. About the unstated explicitly terminology, it will be given in all details. We shall say that the group $G$ is $\Sigma$-cyclic if it is isomorphic to a direct sum of cyclic groups. Likewise, in [12] was established that a group $G$ is $p^{\omega+n}$-projective precisely when there is $P \leq G[p^n]$ with the property that $G/P$ is $\Sigma$-cyclic. Generalizing this concept, in [9] were introduced the following two notions:

- The group $G$ is said to be $n$-simply presented if there exists $P \leq G[p^n]$ with $G/P$ simply presented.
- The group $G$ is said to be strongly (or nicely) $n$-simply presented if there exists a nice subgroup $N \leq G$ with $N \subseteq G[p^n]$ such that $G/N$ is simply presented.

It is self-evident that strongly $n$-simply presented groups are of necessity $n$-simply presented; in [9] a concrete example was constructed showing that the converse is false. Furthermore, it was proved again in [9] that $G$ is $n$-simply presented precisely when it is $n$-co-simply presented, that is, $G \cong$...
The group $G$ is said to be strongly $n$-co-simply presented if $G \cong H/K$ for some simply presented group $H$ and its nice subgroup $K \leq H[p^n]$. 

Unfortunately, an explicit construction from [11] demonstrates that there exists a strongly 1-co-simply presented group of length $\omega + 1$ that is not strongly 1-simply presented. However, for groups of length $\omega$ these two classes coincide with the class of $p^{\omega+n}$-projectives. Even more, each strongly $n$-simply presented group of length $\omega + n$, being $p^{\omega+n}$-projective, is strongly $n$-co-simply presented.

Later on, strengthening the classical notion of total projectivity, in [11] were defined the concepts of $n$-totally projective groups and strongly $n$-totally projective groups as follows:

- The group $G$ is said to be $n$-totally projective if, for every (limit) ordinal $\lambda$, $G/p^\lambda G$ is $p^{\lambda+n}$-projective.
- The group $G$ is said to be strongly $n$-totally projective if, for each (limit) ordinal $\lambda$, $G/p^{\lambda+n}G$ is $p^{\lambda+n}$-projective.

Notice that, when $n = 0$, these groups are just the totally projectives. It is also readily verified that strongly $n$-totally projective groups are $n$-totally projective, whereas the converse implication is not true (cf. [11]). However, it was proved in [10] that $n$-totally projective $A$-groups are themselves strongly $n$-totally projective. (For the full definition of an $A$-group, the reader is referred to [7].)

Likewise, note that (strongly) $n$-simply presented groups are (strongly) $n$-totally projective, respectively.

- The group $G$ is said to be weakly $n$-totally projective if, for each (limit) ordinal $\lambda$, $G/p^{\lambda+n}G$ is $p^{\lambda+2n}$-projective.
- The group $G$ is said to be strong weakly $n$-totally projective if, for every (limit) ordinal $\lambda$, $G/p^{\lambda+n}G$ is $p^{\lambda+2n}$-projective.

It is apparent that the following inclusions hold:

$$\{\text{strongly } n\text{-totally projective}\} \subseteq \{n\text{-totally projective}\} \subseteq \{\text{strong weakly } n\text{-totally projective}\} \subseteq \{\text{weakly } n\text{-totally projective}\}.$$  

Furthermore, in [11] were defined a few more concepts as well. In fact, the above versions of generalizations of simple presentness suggest the following improvements:

- A group $G$ is said to be $m,n$-simply presented if there exists $P \leq G[p^n]$ such that $G/P$ is strongly $m$-simply presented.

In [4] was showed that $G$ is $m,n$-simply presented if and only if there is a strongly $m$-totally projective group $A$ and its $p^n$-bounded subgroup $B$ such that $G \cong A/B$, that is, $G$ is $m,n$-co-simply presented.
A group $G$ is said to be \textit{weakly $m,n$-simply presented} if there exists $N \leq G[p^m]$ such that $N$ is nice in $G$ and $G/N$ is $n$-simply presented.

A very difficult challenging conjecture says that weakly $m,n$-simply presented groups are $m,n$-simply presented, but the most real probability is it to be resolved in the negative. However, for groups of lengths $< \omega^2$ the conjecture holds in the affirmative (see [4]).

- A group $G$ is said to be \textit{m,n-co-weakly simply presented} if there exists an $n$-simply presented group $U$ and its $p^m$-bounded nice subgroup $V$ such that $G \cong U/V$.

Again it is interesting what is the relationship between the classes of $m,n$-co-simply presented groups and $m,n$-co-weakly simply presented groups.

- A group $G$ is said to be \textit{strongly $m,n$-simply presented} if there exists $N \leq G[p^m]$ such that $N$ is nice in $G$ and $G/N$ is strongly $n$-simply presented.

- A group $G$ is said to be \textit{m,n-co-strongly simply presented} if there exists a strongly $n$-simply presented group $X$ and its $p^m$-bounded nice subgroup $Y$ such that $G \cong X/Y$.

A common generalization of both $m,n$-simply presented groups and weakly $m,n$-simply presented groups is the following:

- A group $G$ is said to be \textit{widely $m,n$-simply presented} if there exists $Z \leq G[p^m]$ such that $G/Z$ is $n$-simply presented.

As in [4] a parallel reformulation of $G$ to be widely $m,n$-simply presented is that $G \cong J/Q$, where $J$ is $n$-simply presented and $Q \subseteq J[p^m]$, that is, the group is \textit{widely $m,n$-co-simply presented}.

The alluded to above versions of extensions of total projectivity propose the next further refinements (cf. [11]):

- A group $G$ is said to be \textit{m,$n$-totally projective} if, for any ordinal $\lambda$, $G/p^{\lambda+m}G$ is $p^{\lambda+m+n}$-projective.

Apparently, if $m = 0$, we get $n$-totally projective groups, while if $n = 0$, we obtain strongly $m$-totally projective groups. The combination $m = n = 0$ gives totally projective groups.

Notice also that both $m,n$-simply presented and weakly $m,n$-simply presented groups are themselves $m,n$-totally projective.

Analogously to Propositions 2.1 and 2.2 from [4], and especially similar to the proof of Proposition 2.1, it follows that even widely $m,n$-simply presented groups are $m,n$-totally projective.

Finally, mimicking [3], a group $G$ is termed \textit{nicely $m$-$p^{\omega+n}$-projective} if there exists a $p^m$-bounded nice subgroup $Y$ such that $G/Y$ is $p^{\omega+n}$-projective. More generally, a group $G$ is named \textit{strongly $m$-$\omega_1$-$p^{\omega+n}$-projective} provided that there is a $p^m$-bounded subgroup $T$ such that $G/T$ is strongly $\omega_1$-$p^{\omega+n}$-projective in the sense of [1], that is, a group $A$ is called \textit{strongly $\omega_1$-$p^{\omega+n}$-projective} if there exists a $p^n$-bounded nice subgroup $B$ such that
$G/B$ is the direct sum of a countable group and a $\Sigma$-cyclic group. Note that $p^{\omega+n}$-projectives are obviously strongly $\omega_1$-$p^{\omega+n}$-projective, by taking the countable summand to be zero. Some other interesting definitions of this kind the reader can see in [2].

Our goal here is to introduce certain non-trivial variations of the given above concepts, needed for applicable purposes. Namely, we state the following definitions.

**Definition 1.1.** The group $G$ is called *nicely $m,n$-totally projective* if there is a $p^m$-bounded nice subgroup $N$ such that $G/N$ is $n$-totally projective.

Clearly, if $m = 0$, we obtain $n$-totally projective groups, whereas if $n = 0$, we get strongly $m$-simply presented groups (see [9]). Besides, choosing $m = n = 0$, we also retrieve totally projective (= simply presented) groups.

On the other hand, it is immediate that weakly $m,n$-simply presented group are necessarily nicely $m,n$-totally projective.

**Definition 1.2.** The group $G$ is called *nicely $m,n$-strongly totally projective* if there is a $p^m$-bounded nice subgroup $M$ of $G$ such that $G/M$ is strongly $n$-totally projective.

Observe that nicely $m,n$-strongly totally projective groups are obviously nicely $m,n$-totally projective. Likewise, notice that if $m = 0$, we obtain strongly $n$-totally projective groups, whereas if $n = 0$, we get strongly $m$-simply presented groups (cf. [9]). In particular, if both $m = n = 0$, we just retrieve totally projective (= simply presented) groups.

The last definition can be enlarged to the following one:

**Definition 1.3.** The group $G$ is called *$m,n$-strongly totally projective* if there is a $p^m$-bounded subgroup $P$ of $G$ such that $G/P$ is strongly $n$-totally projective.

Note that $n,m$-simply presented groups are $m,n$-strongly totally projective.

**Definition 1.4.** The group $G$ is called *nicely $m,n$-weakly totally projective* if there is a $p^m$-bounded nice subgroup $X$ of $G$ such that $G/X$ is weakly $n$-totally projective.

**Definition 1.5.** The group $G$ is called *$m,n$-weakly totally projective* if there is a $p^m$-bounded subgroup $Y$ of $G$ such that $G/Y$ is weakly $n$-totally projective.

**Definition 1.6.** The group $G$ is called *nicely $m,n$-strong weakly totally projective* if there is a $p^m$-bounded nice subgroup $K$ of $G$ such that $G/K$ is strong weakly $n$-totally projective.

**Definition 1.7.** The group $G$ is called *$m,n$-strong weakly totally projective* if there is a $p^m$-bounded subgroup $S$ of $G$ such that $G/S$ is strong weakly $n$-totally projective.
Definition 1.8. The group $G$ is called *nicely $m,n$-co-totally projective* if there is an $n$-totally projective group $T$ with a nice $p^m$-bounded subgroup $L$ such that $G \cong T/L$.

Apparently, when $m = 0$, we obtain $n$-totally projective groups, while if $n = 0$, we get strongly $m$-co-simply presented groups (see [9]). If both $m = n = 0$, we come to totally projective (= simply presented) groups.

Definition 1.9. The group $G$ is called *nicely $m,n$-co-strongly totally projective* if there is a strongly $n$-totally projective group $S$ with a nice $p^m$-bounded subgroup $K$ such that $G \cong S/K$.

It is observed that nicely $m,n$-co-strongly totally projective groups are themselves nicely $m,n$-co-totally projective. Also, note that if $m = 0$, we obtain strongly $n$-totally projective groups, while if $n = 0$, we get strongly $m$-co-simply presented groups (cf. [9]). Likewise, the equalities $m = n = 0$ lead to totally projective (= simply presented) groups.

Definition 1.10. The group $G$ is called *nicely $m,n$-co-strongly totally projective* if there is a strongly $n$-totally projective group $H$ with a $p^m$-bounded subgroup $V$ such that $G \cong H/V$.

Definition 1.11. The group $G$ is called *nicely $m,n$-co-weakly totally projective* if there is a weakly $n$-totally projective group $R$ with a $p^m$-bounded nice subgroup $C$ such that $G \cong R/C$.

Definition 1.12. The group $G$ is called *nicely $m,n$-co-strong weakly totally projective* if there is a strong weakly $n$-totally projective group $E$ with a $p^m$-bounded nice subgroup $F$ such that $G \cong E/F$.

Definition 1.13. The group $G$ is called *nicely $m,n$-co-weakly totally projective* if there is a strong weakly $n$-totally projective group $D$ with a $p^m$-bounded subgroup $B$ such that $G \cong D/B$.

In [4] the listed above variations of $m,n$-simply presented groups were characterized, while the main goal here is to characterize the variations of $m,n$-totally projectives defined above by comparing them with the previously cited ones from [4], [9] and [11].

2. Basic Results

We begin with the following statement which determines nicely $m,n$-totally projective groups of length at most $\omega + m$, and which improves Proposition 1.2 from [11].
Theorem 2.1. Suppose that $G$ is a group with $p^{\omega+m}G = \{0\}$. Then $G$ is nicely $m,n$-totally projective if and only if $G$ can be embedded in a $p^{\omega+m}$-bounded $n$-totally projective group.

Proof. “$\Rightarrow$” Assume that $G/N$ is $n$-totally projective for some nice subgroup $N \leq G$ with $p^m N = \{0\}$. Hence $G/N/p^n(G/N) \cong G/(N + p^n G)$ is separable $p^{\omega+n}$-projective. For simpleness we put $N + p^n G = P$. Clearly $P \supseteq p^n G$ remains nice in $G$ because of separability of the above quotient (or because $N$ is nice in $G$), as well as $P \leq G[p^m]$.

On the other hand, let $B$ be a totally projective group whose $p^m B$ is $p^n$-bounded and such that there is an isomorphism $\varphi : p^m B \to P$. Note that there is an abundance of such groups.

Suppose now that $H$ is the group that is the amalgamated sum of $B$ and $G$ along $\varphi$. In other words $H = [B \oplus G]/\{(b, \varphi(b)) : b \in p^m B\}$, i.e., $H = B \oplus G$ where $B \cap G = p^m B = P$.

One may see that $p^n H = p^m B$, so that $H$ will be $p^{\omega+n}$-bounded as well. To that goal, given $x \in p^n H = \cap_{i<\omega}p^i H$ hence $x = b_i + g_i = b_j + g_j = \ldots$ where $b_i \in p^i B$, $b_j \in p^j B$ and $g_i \in p^i G$, $g_j \in p^j G$ for some arbitrary indices $i, j$ with $i < j$. Thus $b_i - b_j = g_j - g_i \in G \cap B = p^n B$ whence $b_i \in p^j B$ for every index $j < \omega$, that is, $b_i \in p^n B = P$. Similarly, $b_j \in p^m B = P$. That is why $g_i \in p^j G + P$ for any $j < \omega$, i.e., $g_i \in \cap_{j<\omega}(p^j G + P) = p^n G + P = P$. Finally, $x \in P = p^m B$, as required.

Furthermore, one can observe that $H/p^m H = (B/p^m B) \oplus (G/P)$, and since $B/p^m B$ is $\Sigma$-cyclic (cf. [5]) while $G/P$ is $p^{\omega+n}$-projective, we deduce that $H/p^m H$ is $p^{\omega+n}$-projective. We finally employ Theorem 4.5 from [9] to get appeared that $H$ is $n$-simply presented. Hence [11] allows us to conclude that $G$ is $n$-totally projective, as stated.

"$\Leftarrow$". Let $G \subseteq H$ where $H$ is an $n$-totally projective group of length not exceeding $\omega + m$. Since $G/(p^m H \cap G) \cong (G + p^m H)/p^m H \subseteq H/p^m H$ is $p^{\omega+n}$-projective as being a subgroup of the $p^{\omega+n}$-projective group $H/p^{\omega+n}H$, and moreover $p^m H \cap G$ is obviously bounded by $p^m$ and is nice in $G$, we establish the wanted claim. \hfill \Box

We next continue with some relationships between the defined above classes of groups.

**Proposition 2.1.** Suppose $G$ is a group. If

(i) $G$ is $\omega_1$-$p^{\omega+m+n}$-projective, then $G$ is widely $m, n$-simply presented.

(ii) $G$ is strongly $\omega_1$-$p^{\omega+m+n}$-projective, then $G$ is $m, n$-simply presented.

**Proof.** (i) In accordance with [8], write $G/H$ is the direct sum of a countable group and a $\Sigma$-cyclic group, whence $G/H$ is simply presented, for some $H \leq G$ with $p^{m+n} H = \{0\}$. Observe that $G/H \cong G/p^m H/H/p^n H$. Therefore, $G/p^n H$ is $n$-simply presented. Since $p^m(p^m H) = \{0\}$, we are finished.
(ii) In virtue of [1], one may write $G/H$ as above into the direct sum of a countable group and a $\Sigma$-cyclic group, but where $H$ is nice in $G$ and $p^{m+n}$-bounded. Furthermore, the same idea as that in point (i) works, seeing that $H/p^n H$ remains nice in $G/p^n H$ and hence $G/p^n H$ is strongly $n$-simply presented. 

\begin{proposition}
Let $G$ be a nicely $m, n$-strongly totally projective group such that $p^{\omega+m+n} G = \{0\}$. Then $G$ is nicely $m-p^{\omega+n}$-projective.
\end{proposition}

\begin{proof}
Assume that $G/M$ is strongly $n$-totally projective for some nice $p^m$-bounded subgroup $M$. Utilizing [11], the quotient $G/M/p^{\omega+n}(G/M) \cong G/(M + p^{\omega+n}G)$ is $p^{\omega+n}$-projective. Since $p^n(M + p^{\omega+n}G) = \{0\}$, and $M + p^{\omega+n}G$ remains nice in $G$, the result follows. 
\end{proof}

With the last statement in hand, one may derive the following:

\begin{theorem}
Suppose that $G$ is a group with countable $p^{\omega+m+n} G$. Then $G$ is nicely $m, n$-strongly totally projective if and only if $G$ is strongly $m-\omega_1-p^{\omega+n}$-projective.
\end{theorem}

\begin{proof}
"$\Rightarrow$" Appealing to Proposition 3.1 (ii), stated and proved below, the factor-group $G/p^{\omega+m+n}G$ is also nicely $m, n$-strongly totally projective. Furthermore, Proposition 2.2 is applicable to get that $G/p^{\omega+m+n}G$ is nicely $m-p^{\omega+n}$-projective and hence strongly $m-\omega_1-p^{\omega+n}$-projective. Since $p^{\omega+m+n}G$ is countable by assumption, we employ Theorem 3.11 from [3] to deduce the desired implication.

"$\Leftarrow$" It follows immediately because strongly $\omega_1-p^{\omega+n}$-projective groups are themselves strongly $n$-simply presented (see [1]) and so they are strongly $n$-totally projective. 
\end{proof}

\begin{proposition}
If $G$ is a nicely $m, n$-totally projective group of length $\lambda < \omega^2$, then $G$ is weakly $m, n$-simply presented, and vice versa, provided $\text{length}(G) < \omega^2$.
\end{proposition}

\begin{proof}
Suppose that $G$ is a nicely $m, n$-totally projective group. Thus $G/N$ is $n$-totally projective for some nice subgroup $N$ of $G$ which is bounded by $p^m$. Since $p^\lambda(G/N) = (p^\lambda G + N)/N = \{0\}$, we may apply [11] to get that $G/N$ is $n$-simply presented, as required.

The converse implication is elementary. 
\end{proof}

As a consequence, we yield:

\begin{corollary}
If $G$ is a nicely $m, n$-totally projective group of length $< \omega^2$, then $G$ is $m, n$-simply presented (and, in particular, is $n, m$-strongly totally projective).
\end{corollary}

The same can be said adding the word "strongly". Specifically, the following is valid:
**Proposition 2.4.** If $G$ is a nicely $m,n$-strongly totally projective group of length $\lambda < \omega^2$, then $G$ is strongly $m,n$-simply presented, and visa versa, provided $\text{length}(G) < \omega^2$.

**Proof.** Utilizing the corresponding definitions, the same idea as that in Proposition 2.3 works. \hfill \square

Similarly, we derive:

**Proposition 2.5.** Suppose that $G$ is a group of length strictly less than $\omega^2$. Then $G$ is nicely $m,n$-co-totally projective if and only if $G$ is $m,n$-co-weakly simply presented.

**Proposition 2.6.** If the group $G$ is either

(a) nicely $m,n$-totally projective, or

(b) nicely $m,n$-co-totally projective,

then $G$ is $m,n$-totally projective.

**Proof.** (a) Assume that there exists a nice $p^m$-bounded subgroup $N$ of $G$ such that $G/N$ is $n$-totally projective. Since we have the isomorphism sequence

$$G/N/p^\lambda(G/N) = G/N/(p^\lambda G + N)/N \cong G/(p^\lambda G + N) \cong G/p^{\lambda + m} G/(p^\lambda G + N)/p^{\lambda + m} G$$

where $G/N/p^\lambda(G/N)$ is $p^{\lambda + n}$-projective for each limit ordinal $\lambda$ and $(p^\lambda G + N)/p^{\lambda + m} G$ is $p^m$-bounded, we apply [11] to infer that $G/p^{\lambda + m} G$ is $p^{\lambda + m + n}$-projective, as required.

(b) Assume that there exists an $n$-totally projective group $T$ with a $p^m$-bounded nice subgroup $L$ such that $G \cong T/L$. Furthermore, we deduce that

$$G/p^{\lambda + m} G \cong T/L/p^{\lambda + m}(T/L)$$

$$\cong T/L/(p^{\lambda + m} T + L)/L$$

$$\cong T/(p^{\lambda + m} T + L).$$

But

$$T/p^\lambda T \cong T/(p^{\lambda + m} T + L)/p^\lambda T/(p^{\lambda + m} T + L)$$

is $p^{\lambda + n}$-projective for every limit ordinal $\lambda$ and $p^{\lambda} T/(p^{\lambda + m} T + L)$ is $p^m$-bounded, so we employ [11] to conclude that $T/(p^{\lambda + m} T + L) \cong G/p^{\lambda + m} G$ is $p^{\lambda + m + n}$-projective, as requested.

Note that the condition $p^m L = \{0\}$ was not utilized. \hfill \square

**Remark 1.** For some subclasses of groups of these alluded to above, we refer to [4].

For $p^\omega$-bounded groups, we can say even a little more. Especially the following is true (compare with Theorem 2.5 of [4]):
**Theorem 2.3.** Suppose that $G$ is a group with $p^\omega G = \{0\}$. Then the following conditions are equivalent:

(i) $G$ is $m,n$-totally projective;

(ii) $G$ is nicely $m,n$-totally projective;

(iii) $G$ is nicely $m,n$-co-totally projective;

(iv) $G$ is $p^{\omega+m+n}$-projective.

**Proof.** The equivalence (i) $\iff$ (iv) was proved in [11]. What remains to show is that (iv) implies both (iii) and (ii). In fact, since $G$ is $p^{\omega+m+n}$-projective, $G \cong S/Y$ for some $\Sigma$-cyclic group $S$ with a $p^{m+n}$-bounded subgroup $Y$. Put $X = S[p^n] \cap Y = Y[p^n]$. Thus $X$ is nice in $S$ as the intersection of two closed subgroups (see, for example, [5]). Furthermore, $G \cong S/X/Y/X$, where $S/X$ is obviously $p^{\omega+n}$-projective because $p^nX = \{0\}$, and hence $S/X$ is strongly $n$-simply presented. But $Y/X = Y/Y[p^n] \cong p^n Y$ is bounded by $p^m$ and is also nice in $S/X$ taking into account that $G$ is separable, so that $Y$ is nice in $S$ (cf. [5]). Now, an appeal to Definition 1.3 gives that $G$ is nicely $m,n$-co-totally projective.

As for the second implication, since $G$ is $p^{\omega+m+n}$-projective, there is $V \leq G[p^{m+n}]$ such that $G/V$ is $\Sigma$-cyclic. Set $U = G[p^n] \cap V = V[p^n]$. Hence $U$ is nice in $G$ as the intersection of two closed subgroups (see, for instance, [5]). Moreover, $G/U/V/U \cong G/V$ is $\Sigma$-cyclic with $V/U = V/V[p^n] \cong p^n V$ being bounded by $p^n$. Consequently, $G/U$ is $p^{\omega+n}$-projective, whence $n$-totally projective, with $p^n U = \{0\}$. With Definition 1.1 at hand, this guarantees that $G$ is nicely $m,n$-totally projective, as stated.

The next example demonstrates that beyond lengths $\omega$, the last result is not longer valid, and also that the concept of $m,n$-totally projective groups is independent of that of nicely $m,n$-totally projective groups – the same can be happen for nicely $m,n$-co-totally projective groups (see [4] too).

**Example 2.1.** There exists a $p^{\omega+1}$-bounded 1,1-totally projective group which is not nicely 1,1-totally projective.

**Proof.** We begin with the following:

**Claim 1.** Let $H$ be a $p^{\omega+1}$-projective group, and let $J$ be a countable subgroup of $H$. Then $p\overline{J}$ is countable.

To show this, if $P$ is a $p$-bounded subgroup of $H$ such that $H/P$ is $\Sigma$-cyclic, then there is a subgroup $L$ of $H$ containing $P$ and $J$ such that $L/P$ is a countable of $H/P$. It follows that $L$ is closed in $H$, so that $\overline{J} \subseteq L$. Since $L = P + X$ for some countable subgroup $X$, we have $p\overline{J} \subseteq pL = pX$ is countable.

**Claim 2.** Let $B$ be the standard separable free valuated vector space (i.e., all its finite Ulm-Kaplansky invariants equal to 1). Then there is a subspace $V \subseteq \overline{B}$ of uncountable rank, containing $B$, such that if $C$ is any closed subspace of $\overline{B}$ contained in $V$, then $C(k) = C \cap \overline{B}(k) = \{0\}$ for some $k < \omega$.
(i.e., any closed subspace of $\overline{B}$ - which, in fact, will be a valued direct summand - contained in $V$ is bounded).

Let $b_i$ for $i < \omega$ be a basis for $B$. Let $C_\alpha$ for $\alpha < c = 2^{\aleph_0}$ be a list of all the unbounded closed subspaces of $\overline{B}$; note that each $C_\alpha$ has rank $c$. Construct elements $x_\alpha$ and $y_\alpha$ for $\alpha < c$ such that (1) $y_\alpha \in C_\alpha$, and (2) $\{b_i, x_\alpha, y_\alpha : i < \omega, \alpha < c\}$ is linearly independent. If we let $V = \text{span}\{b_i, x_\alpha, y_\alpha : i < \omega, \alpha < c\}$, then for any unbounded closed subspace $C_\alpha$ of $\overline{B}$, we have $y_\alpha \in C_\alpha \setminus V$, which shows that $C_\alpha$ is not contained in $V$.

Consider $V \subseteq \overline{B}$ as in Claim 2. Let $Y$ be a separable group such that $Y[p]$ is isometric to $V$. Let $Y_1$ be a group with $Y_1[p] = Y[p]$ and $Y = pY_1 \cong Y_1/Y_1[p]$. If $C_1$ is the torsion completion of $Y_1$, then $C = pC_1 \cong C_1/C_1[p]$ is the torsion completion of $Y$. Let $P$ be the valued group

$$\left( \frac{C_1}{Y_1[p]} \right)[p^2] = \left( \frac{Y_1[p^3]}{C_1[p^2]} \right)/Y_1[p].$$

We can identify $Y[p^2] \cong Y_1[p^3]/Y_1[p]$ with a subgroup of $P$. In addition,

$$P[p] \cong (Y_1[p^2]/Y_1[p]) \oplus (C_1[p]/Y_1[p]) \cong Y[p] \oplus (C_1[p]/Y_1[p]).$$

We will be done if we can show the following:

**Claim 3.** Suppose $G$ is a group containing $P$ such that the valuation on $P$ agrees with the height function on $G$, and so that $G/P$ is $\Sigma$-cyclic. Then $G$ is 1, 1-simply presented of length $\omega + 1$, and hence it is 1, 1-totally projective of the same length, but $G$ is not weakly 1, 1-simply presented; even more, $G \oplus X$ is not weakly 1, 1-simply presented for every $\Sigma$-cyclic group $X$. By virtue of Proposition 2.3, this means that it is not nicely 1, 1-totally projective.

To this aim, suppose $M$ is a nice $p$-bounded subgroup of $G$ such that $G/M$ is 1-simply presented. Note that $M + p^{\omega}G$ will also be nice in $G$ and $p$-bounded, and $G/[M + p^{\omega}G] \cong G/M + p^{\omega}(G/M)$ will be $p^{\omega + 1}$-projective, and so 1-simply presented. So, we may assume $p^{\omega}G \subseteq M$.

Since $M$ is nice, $M/p^{\omega}G$ will be closed in $(G/p^{\omega}G)[p]$. Consider $M' = \left( \frac{M}{p^{\omega}G} \right) \cap (P/P(\omega))[p]$; so $M'$ is closed in $(P/P(\omega))[p] \cong C[p]$. Observe $M' \subseteq Y[p] = V$, and moreover it follows from Claim 2 that $M'$ is bounded. In other words, for some integer $k$, we must have $M' \cap V(k) = \{0\}$.

Let $Z$ be a basic subgroup of $Y$ and let $Z = Z'_k \oplus Z_k$ be a decomposition, where $Z'_k$ is a maximal $p^k$-bounded summand of $Z$. This determines a decomposition $Y = Z_k \ominus Y_k$ of $Y$.

Notice that $Y_k[p^2] \cap M = \{0\}$, so that it embeds isomorphically in $G/M$. Call this image $L$ and let $J \subseteq L$ be the image of $Z_k[p^2] \subseteq Y_k[p^2] \subseteq G$ in $G/M$. Note that $J$ is countable, and since $Z_k[p^2]$ is dense in $Y_k[p^2]$, it follows that $J$ is dense in $L$. However, since $pL \cong pY_k$ is uncountable, we obtain that $p\overline{J}$ is also uncountable. But this contradicts Claim 1, and thus proves our assertion after all.

$\square$
The next question arises quite naturally: Does there exist a \( p^{\omega+1} \)-bounded 1,1-totally projective group that is not nicely 1,1-co-totally projective? Even more, in view of Proposition 2.6, is there a nicely 1,1-totally projective group which is not nicely 1,1-co-totally projective?

However the converse to that question is true for the “strongly” situation.

**Example 2.2.** There exists a nicely 1,1-co-totally projective group of length \( \omega + 1 \) which is not nicely 1,1-strongly totally projective.

**Proof.** As already mentioned before, in Example 2.1 from [9] was constructed a \( p^{\omega+1} \)-bounded strongly 1-co-simply presented group which is not strongly 1-simply presented. We furthermore wish apply Theorem 3.2 of [11] to get the desired claim. □

Recall that it was defined in [8] a group \( G \) to be \( \omega_1 \cdot p^{\omega+n} \)-projective, provided that there exists a countable (nice) subgroup \( C \) such that \( G/C \) is \( p^{\omega+n} \)-projective.

In the light of the last constructions, we obtain the following strengthening of Theorem 2.3:

**Proposition 2.7.** Suppose that \( G \) is a group with countable \( p^{\omega+m}G \). Then \( G \) is \( m,n \)-totally projective if and only if \( G \) is \( \omega_1 \cdot p^{\omega+m+n} \)-projective.

**Proof.** “Necessity”: Accordingly, \( G/p^{\omega+m}G \) is \( p^{\omega+m+n} \)-projective. We therefore see that the above definition from [8] works to get the assertion.

“Sufficiency”: It follows directly from Proposition 2.1 (i) stated and proved above. □

### 3. Ulm subgroups and Ulm factors

Imitating [5] and/or [6], for any group \( G \) and any \( n \in \mathbb{N} \), we define \( p^nG = \{ p^n g \mid g \in G \} \). Set \( p^\alpha G = \cap_{\beta < \alpha} p^\beta G \) whenever \( \alpha \) is limit, whereas \( p^\alpha G = p(p^{\alpha-1}G) \) provided that \( \alpha \) is nonlimit. Clearly \( p^\alpha G \leq G \) and these subgroups are called Ulm subgroups, while the factor-groups \( G/p^\alpha G \) are said to be Ulm factors.

We will now study Nunke’s type results for the new group classes.

**Proposition 3.1.** (i) If \( G \) is nicely \( m,n \)-totally projective, then so are \( p^\alpha G \) and \( G/p^\alpha G \) for any ordinal \( \alpha \).

(ii) If \( G \) is nicely \( m,n \)-strongly totally projective, then so are \( p^\alpha G \) and \( G/p^\alpha G \) for any ordinal \( \alpha \).

**Proof.** (i) Let \( p^mN = \{ 0 \} \) where \( N \) is nice in \( G \) such that \( G/N \) is \( n \)-totally projective. Clearly \( N \cap p^\alpha G \) is \( p^m \)-bounded and nice in \( p^\alpha G \) (see [5]) as well as \( p^\alpha G/(p^\alpha G \cap N) \cong (p^\alpha G + N)/N = p^\alpha (G/N) \) is \( n \)-totally projective because the same is \( G/N \) (cf. [11]), thus proving the first half.
For the other part, \((N + p^\alpha G)/p^\alpha G\) is \(p^m\)-bounded and nice in \(G/p^\alpha G\) (cf. [5]). Also,
\[
G/p^\alpha G/(N + p^\alpha G)/p^\alpha G \cong G/(N + p^\alpha G) \cong G/N/(N + p^\alpha G)/N = G/N/p^\alpha(G/N)
\]
is \(n\)-totally projective since so is \(G/N\) (see [11]), thus showing the second half.

(ii) Follows by similar arguments seeing that \(p^\alpha(G/N)\) and \(G/N/p^\alpha(G/N)\) are both strongly \(n\)-totally projective, provided that \(G/N\) is so (cf. [11]).

\[\square\]

**Proposition 3.2.**

(i) If \(G\) is nicely \(m, n\)-co-totally projective, then the same are \(p^\alpha G\) and \(G/p^\alpha G\) for any ordinal \(\alpha\).

(ii) If \(G\) is nicely \(m, n\)-co-strongly totally projective, then the same are \(p^\alpha G\) and \(G/p^\alpha G\) for any ordinal \(\alpha\).

**Proof.** (i) Let \(G \cong T/L\) for some \(n\)-totally projective group \(T\) with a \(p^m\)-bounded nice subgroup \(L\). Hence \(p^\alpha G \cong p^\alpha(T/L) = (p^\alpha T + L)/L \cong p^\alpha T/(p^\alpha T \cap L)\), with \(n\)-totally projective \(p^\alpha T\) (see [11]) and \(p^\alpha T \cap L\) being \(p^m\)-bounded and nice in \(p^\alpha T\) (cf. [5]). This shows that \(p^\alpha G\) is nicely \(m, n\)-co-totally projective.

Furthermore, concerning the second part-half, \(G/p^\alpha G \cong T/L/p^\alpha(T/L) = T/L/(p^\alpha T + L)/L \cong T/(p^\alpha T + L) \cong T/p^\alpha T/(p^\alpha T + L)/p^\alpha T\). The utilization of [11] ensures that \(T/p^\alpha T\) is \(n\)-totally projective. Moreover, \((p^\alpha T + L)/p^\alpha T \cong L/(p^\alpha T \cap L)\) is \(p^m\)-bounded and nice in \(T/p^\alpha T\) because \(p^\alpha T + L\) is so in \(T\) (cf. [5]). This guarantees that \(G/p^\alpha G\) is nicely \(m, n\)-co-totally projective.

(ii) Follows via identical arguments as above, observing that \(T\) being strongly \(n\)-totally projective implies the same for both \(p^\alpha T\) and \(T/p^\alpha T\) (see [11]).

We now have all the ingredients needed to prove the following assertion. It reduces the study of nicely \(m, n\)-strong total projectivity to Ulm subgroups and Ulm factors.

**Theorem 3.1.** Suppose that \(\alpha\) is an ordinal. Then the group \(G\) is nicely \(m, n\)-strongly totally projective iff both \(p^{\alpha+m+n}G\) and \(G/p^{\alpha+m+n}G\) are nicely \(m, n\)-strongly totally projective.

**Proof.** The necessity follows from Proposition 3.1 (ii), replacing \(\alpha\) by \(\alpha + m + n\).

Concerning the sufficiency, denote \(k = m + n\). With Definition 1.2 at hand, let us assume that \(p^{\alpha+k}G/H = p^{\alpha+k}(G/H)\) is strongly \(n\)-totally projective for some \(p^m\)-bounded nice subgroup \(H\) of \(p^{\alpha+k}G\). Thus \(H\) is nice in \(G\) as well (see [5]).

Also, suppose \(G/p^{\alpha+k}G/A/p^{\alpha+k}G \cong G/A\) is strongly \(n\)-totally projective for some \(A \leq G\) such that \(A/p^{\alpha+k}G\) is nice in \(G/p^{\alpha+k}G\) and \(p^m A \subseteq p^{\alpha+k}G\). Therefore, \(A\) is nice in \(G\) too (cf. [5]).
We will now use a trick used in [4], [9] and [11], respectively. Let $V$ be a maximal $p^m$-bounded summand of $p^{\alpha+n}G$; so there exists a decomposition $p^{\alpha+n}G = U \oplus V$ for some $U \leq p^{\alpha+n}G$. Besides, let $K$ be a $p^{\alpha+k}$-high subgroup of $G$ containing $V$. Now, it follows that (see, for instance, [9] and [11])

$$(G/p^{\alpha+k}G)[p^m] = (U \oplus K[p^m])/p^{\alpha+k}G,$$

whence $A \subseteq U \oplus K[p^m]$. Therefore, $U + A \subseteq U \oplus K[p^m]$ and hence the modular law from [5] yields $U + A = (U \oplus K[p^m]) \cap (U + A) = U + (U + A) \cap K[p^m]$. Letting $(U + A) \cap K[p^m] = B$, we deduce that $U + A = U + B$ with $p^mB = \{0\}$. Since $U \subseteq p^{\alpha+n}G \subseteq p^\alpha G$, we have that $p^{\alpha+n}G + A = p^{\alpha+n}G + B$.

Next put $Z = B + H$. By what we have already established above, it follows that $p^mZ = \{0\}$ and that $p^{\alpha+n}G + Z = p^{\alpha+n}G + B = p^{\alpha+n}G + A$. Furthermore, $A$ being nice in $G$ elementary insures that $p^{\alpha+n}G + Z = p^{\alpha+n}G + A$ is nice in $G$ as well. Moreover, the modular law ensures that $p^{\alpha+k}G \cap Z = p^{\alpha+k}G \cap (B + H) = p^{\alpha+k}G \cap B + H = p^{\alpha+k}G \cap K[p^m] \cap (U + A) + H = H$ is nice in $p^{\alpha+k}G$. Applying Lemma 2.9 from [4], we conclude that $p^{\alpha+n}G \cap Z$ is nice in $p^{\alpha+n}G$, and hence in $G$ (cf. [5]), because $k \geq n$. Finally, we again employ [5] to get that after all $Z$ is, in fact, nice in $G$.

On the other hand, using the niceness of $Z$ in $G$, we derive that $p^{\alpha+k}(G/Z) = (p^{\alpha+k}G + Z)/Z \cong p^{\alpha+k}G/(p^{\alpha+k}G \cap Z) = p^{\alpha+k}G/H$ is strongly $n$-totally projective. So, [11] applies to infer that $p^{\alpha+n}(G/Z)$ is strongly $n$-totally projective since $k \geq n$. In virtue again of ([11], Theorem 2.5), $G/Z/p^{\alpha+n}(G/Z) = G/Z/(p^{\alpha+n}G + Z)/Z \cong G/(p^{\alpha+n}G + Z) = G/(p^{\alpha+n}G + A) \cong G/A/(p^{\alpha+n}G + A)/A = G/A/p^{\alpha+n}(G/A)$ is strongly $n$-totally projective, too. We once again employ ([11], Corollary 2.8) to detect that $G/Z$ is strongly $n$-totally projective, as wanted. \hfill \Box

**Remark 2.** It seems that $k = m + n$ cannot be minimized to $m$ or $n$ as it was done in [4].

## 4. Left-open Problems

In closing we pose the following list of still unsettled questions and conjectures.

**Question 3.1.** Suppose $G$ is a group such that $G/p^\lambda G$ is totally projective for some ordinal $\lambda$. Is then $G$ nicely $m, n$-totally projective if and only if $p^\lambda G$ is?

**Question 3.2.** Suppose $G$ is a group such that $G/p^\lambda G$ is totally projective for some ordinal $\lambda$. Is then $G$ nicely $m, n$-strongly totally projective if and only if $p^\lambda G$ is?

These questions will have a positive solution provided the following implication holds: If $A$ is a group such that $p^\lambda A$ is $n$-totally projective and $A/p^\lambda A$ is totally projective, then $A$ is $n$-totally projective.
In regard to Corollary 2.1, one can state the following:

**Question 3.3.** If \( G \) is a nicely \( m,n \)-totally projective group, is then \( G \) an \( n,m \)-strongly totally projective group?

**Conjecture 3.1.** Every \( n \)-simply presented group is a summand of a strongly \( n \)-simply presented group; in particular, for any \( n \), there is an \( n \)-simply presented group which is not strongly \( n \)-simply presented.

Same for the co-case.

**Conjecture 3.2.** For any \( n \geq 0 \), there exists a strongly \( n \)-simply presented group of length \( \omega + n + 1 \) that is not strongly \( n \)-co-simply presented.

As noted above, the definition of an A-group is stated in [7].

**Conjecture 3.3.** Let \( G \) be an A-group. Then \( G \) is \( n \)-simply presented if and only if \( G \) is strongly \( n \)-simply presented.

Same for the co-case.

Since as aforementioned \( G \) is \( n \)-simply presented exactly when it is \( n \)-co-simply presented, if the last conjecture is true one may derive that \( G \) is strongly \( n \)-simply presented uniquely when it is strongly \( n \)-co-simply presented, provided \( G \) is an A-group.

**Conjecture 3.4.** Suppose \( G \) is an A-group. Then \( G \) is weakly \( n \)-totally projective if and only if \( G \) is strong weakly \( n \)-totally projective.

Thus, since it was demonstrated in [10] that there exists a weakly \( n \)-totally projective A-group which is not \( n \)-totally projective, if this conjecture holds in the affirmative, we will have an example of a strong weakly \( n \)-totally projective A-group that is not \( n \)-totally projective.

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**References**


