On Fuzzy Differential Subordination

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Abstract. The theory of differential subordination was introduced by S.S.Miller and P.T.Mocanu in [2], then developed in many papers. In [1] the authors investigate various subordination results for some subclasses of analytic functions in the unit disc. G.I.Oros and G.Oros define the notion of fuzzy subordination and in [3, 4, 5] they define the notion of fuzzy differential subordination. In this paper, we determine sufficient conditions for a multivalent function to be a dominant of the fuzzy differential subordination.

1. Introduction

We introduce some basic notions and results that are used in the sequel.

Definition 1.1 ([6]). Let $X$ be a non-empty set. An application $F : X \rightarrow [0, 1]$ is called fuzzy subset. An alternate definition, more precise, would be the following: A pair $(A, F_A)$, where $F_A : X \rightarrow [0, 1]$ and

$$A = \{x \in X : 0 < F_A(x) \leq 1\} = supp(A, F_A),$$

is called fuzzy subset.

Proposition 1.1 ([3]). If $(M, F_M) = (N, F_N)$, then we have $M = N$, where $M = supp(M, F_M), N = supp(N, F_N)$.

Proposition 1.2 ([3]). If $(M, F_M) \subseteq (N, F_N)$, then we have $M \subseteq N$, where $M = supp(M, F_M), N = supp(N, F_N)$.

We also need the following notations and results from the classical complex analysis [5].

For $D \subseteq \mathbb{C}$, we denote by $\mathcal{H}(D)$ the class of holomorphic functions on $D$, and by $\mathcal{H}_n(D)$ the class of holomorphic and univalent functions on $D$.

In this paper, we denote by $\mathcal{H}(U)$ the set of holomorphic functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with $\partial U = \{z \in \mathbb{C} : |z| = 1\}$ the boundary of the unit disc.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we denote
\[ \mathcal{H}[a, n] = \{ f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U \}, \]
\[ A_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots, z \in U \} \text{ with } A_1 = A, \]
and \( S = \{ f \in A : f \text{ a univalent function in } U \}. \)

Let \( \mathcal{B} = \{ \varphi \in \mathcal{H}(U) : \varphi(0) = 0, |\varphi(z)| < 1, z \in U \} \) denote the class of Schwarz functions.

**Definition 1.2** ([4]). Let \( f, g \in \mathcal{H}(U) \). We say that the function \( f \) is subordinated to \( g \), written \( f < g \) or \( f(z) < g(z) \) if there exists a function \( w \in \mathcal{H}(U) \) with \( w(0) = 0 \) and \( |w(z)| < 1, z \in U \) (which means \( w \in \mathcal{B} \)) such that \( f(z) = g(w(z)), z \in U \).

Let \( D \subset \mathbb{C} \) and \( f, g \in \mathcal{H}(D) \) holomorphic functions. We denote by
\[
 f(D) = \{ f(z) | 0 < F_{f(D)} f(z) \leq 1, z \in D \} = \supp(f(D), F_{f(D)})
\]
and
\[
 g(D) = \{ g(z) | 0 < F_{g(D)} g(z) \leq 1, z \in D \} = \supp(g(D), F_{g(D)}).
\]

**Definition 1.3** ([5]). Let \( D \subset \mathbb{C} \), \( z_0 \in D \) be a fixed point, and let the functions \( f, g \in \mathcal{H}(D) \). The function \( f \) is said to be fuzzy subordinate to \( g \) and write \( f < \_\_ g \) or \( f(z) < \_\_ g(z) \), if
1. \( f(z_0) = g(z_0) \),
2. \( F_{f(D)} f(z) \leq F_{g(D)} g(z), z \in D. \)

**Proposition 1.3** ([5]). Let \( D \subset \mathbb{C} \), \( z_0 \in D \) be a fixed point, and let the functions \( f, g \in \mathcal{H}(D) \). If \( f(z) < \_\_ g(z), z \in D \), then
1. \( f(z_0) = g(z_0) \),
2. \( f(D) \subseteq g(D) \), where \( f(D) = \supp(f(D), F_{f(D)}), g(D) = \supp(g(D), F_{g(D)}). \)

The equality occurs if and only if \( F_{f(D)} f(z) = F_{g(D)} g(z) \). Denoted by
\[
 S^* = \{ f \in A : \text{Re} \frac{zf'(z)}{f(z)} > 0, z \in U \}
\]
the class of normalized starlike functions in \( U \),
\[
 K = \{ f \in A : \text{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \}
\]
the class of normalized convex functions in \( U \) and by
\[
 C = \{ f \in A : \exists \varphi \in K, \text{Re} \frac{f(z)}{\varphi(z)} > 0, z \in U \}
\]
the class of normalized close-to-convex functions in \( U \) [5].

Let \( J(\alpha, f; z) = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha (1 + \frac{zf''(z)}{f'(z)}), z \in U, \) for \( \alpha \) real number and \( f \in A_p \) [2].

Let \( \Omega = \supp(\Omega, F_\Omega) = \{ z \in \mathbb{C} : 0 < F_\Omega(z) \leq 1 \}, \)
\( \Delta = \supp(\Delta, F_\Delta) = \{ z \in \mathbb{C} : 0 < F_\Delta(z) \leq 1 \}, p(U) = \supp(p(U), F_{p(U)}) \)
\( = \{ f(z) : 0 < F_{p(U)}(f(z)) \leq 1, z \in U \} \) and
\( \psi(C^3 \times U) = \supp(\psi(C^3 \times U), F_{\psi(C^3 \times U)}) \)
\( = \{ \psi(p(z), zp^2 p''(z); z) : 0 < F_{\psi(C^3 \times U)}(\psi(p(z), zp^2 p''(z), z)) \leq 1, z \in U \} \)
[4].
Theorem 1.3 implies where is analytic in \( p \in \mathbb{C}^3 \times U \to \mathbb{C} \) and let \( h \) be univalent in \( U \). If \( p \) is analytic in \( U \) and satisfies the (second-order) fuzzy differential subordination

\[
F_{\psi(C^3 \times U)}(\psi(p(z), zp'(z), z^2p''(z); z) \leq F_{h(U)}h(z) \tag{1}
\]

i.e. \( \psi(p(z), zp'(z), z^2p''(z); z) \prec_F h(z), z \in U \), then \( p \) is called a fuzzy solution of the fuzzy differential subordination. The univalent function \( h \) containing \( p \) is analytic in \( U \) and satisfies the (second-order) fuzzy differential subordination, or more simple a fuzzy dominant, if \( p(z) \prec_F g(z), z \in U \), for all \( p \) satisfying (1). A fuzzy dominant \( \tilde{q} \) that satisfies \( \tilde{q}(z) <_F q(z), z \in U \), for all fuzzy dominant \( q \) of (1) is said to be the fuzzy best dominant of (1).

**Theorem 1.1** ([5]). Let \( h \) be analytic in \( U \), let \( \phi \) be analytic in domain \( D \) containing \( h(U) \) and suppose

a) \( Re[\phi(h(z))] > 0, z \in U \) and

b) \( h(z) \) is convex.

If \( p \) is analytic in \( U \), with \( p(0) = h(0), p(U) \subset D \) and \( \psi(C^2 \times U) \to \mathbb{C}, \psi(p(z), zp'(z)) = p(z) + zp'(z) \), \( \phi[p(z)] \) is analytic in \( U \), then

\[
F_{\psi(C^2 \times U)}\psi(p(z), zp'(z)) \leq F_{h(U)}h(z),
\]

implies

\[
F_{p(U)}p(z) \leq F_{h(U)}h(z), z \in U,
\]

where

\[
\psi(C^2 \times U) = supp(C^2 \times U, F_{\psi(C^2 \times U)}\psi(p(z), zp'(z)) = \{ z \in \mathbb{C} : 0 < F_{\psi(C^2 \times U)}\psi(p(z), zp'(z)) \leq 1 \},
\]

\[
h(U) = supp(U, F_{h(U)}h(z)) = \{ z \in \mathbb{C} : 0 < F_{h(U)}h(z) \leq 1 \}.
\]

**Theorem 1.2** ([5]). Let \( h \) be convex in \( U \) and let \( P : U \to \mathbb{C} \), with \( ReP(z) > 0 \). If \( p \) is analytic in \( U \) and \( \psi : C^2 \times U \to \mathbb{C}, \psi(p(z), zp'(z)) = p(z) + P(z)zp'(z) \) is analytic in \( U \), then

\[
F_{\psi(C^2 \times U)}[p(z) + P(z)zp'(z)] \leq F_{h(U)}h(z),
\]

implies

\[
F_{p(U)}P(z) \leq F_{h(U)}h(z), z \in U.
\]

**Theorem 1.3** ([5]). (Hallenbeck and Ruscheweyh) Let \( h \) be a convex function with \( h(0) = a \), and let \( \gamma \in \mathbb{C}^* \) be a complex number with \( Re\gamma \geq 0 \). If \( p \in H[a, n] \) with \( p(0) = a \) and \( \psi : C^2 \times U \to \mathbb{C}, \psi(p(z) + zp'(z)) = p(z) + \frac{1}{\gamma}zp'(z) \) is analytic in \( U \), then

\[
F_{\psi(C^2 \times U)}[p(z) + \frac{1}{\gamma}zp'(z)] \leq F_{h(U)}h(z),
\]

implies

\[
F_{p(U)}p(z) \leq F_{q(U)}q(z) \leq F_{h(U)}h(z), z \in U,
\]
Proposition 2.2.\Proof

The proof of Proposition is similar to Theorem 1.1[5].

In addition, assume that

(iii) \(\Re(\frac{zh'(z)}{Q(z)}) = \Re(\frac{q'(z)}{q(z)}) + \frac{1}{z^2} > 0\).

If \(p\) is analytic in \(U\), with \(p(0) = q(0), p(U) \subset D\) and \(\psi : \mathbb{C}^2 \times U \to \mathbb{C}\),
\(\psi(p(z), zp'(z)) = p(z) + zp'(z)\cdot\phi(p(z))\) is analytic in \(U\), then
\[ F_{\psi(C^2 \times U)}[p(z) + zp'(z)\cdot\phi(p(z))] \leq F_{h(U)}h(z), \]

implies
\[ F_{p(U)}p(z) \leq F_{q(U)}q(z), z \in U, \text{ i.e.} \]
\(p(z) <_{F} q(z)\), and \(q\) is the best dominant, where
\[ \psi(C^2 \times U) = \supp(C^2 \times U, F_{\psi(C^2 \times U)}\psi(p(z), zp'(z))) \]
\(= \{z \in \mathbb{C} : 0 < F_{\psi(C^2 \times U)}\psi(p(z), zp'(z)) \leq 1\},\)
and
\[ h(U) = \supp(U, F_{h(U)}h(z)) = \{z \in \mathbb{C} : 0 < F_{h(U)}h(z) \leq 1\}. \]

\Proof The proof of Proposition is similar to Theorem 1.1[5]. \qed

Proposition 2.2. Let \(q \in \mathcal{H}[p, p]\) be univalent, \(q(z) \neq 0\) and satisfies the following conditions.

(i) \(\frac{zd'(z)}{q(z)}\) is starlike,

(ii) \(\Re(\frac{z\alpha}{Q(z)} + 1 + \frac{zq''(z)}{q(z)} - \frac{zq'(z)}{q(z)}) > 0\) for all \(\alpha \neq 0\) and for all \(z \in U\).

For \(p \in \mathcal{H}[p, p]\) with \(p(z) \neq 0\) in \(U\) and
\[ \psi : \mathbb{C}^2 \times U \to \mathbb{C}, \psi p(z), zp'(z)) = p(z) + \alpha \frac{zp'(z)}{p(z)} \]
is analytic in \(U\), then
\[ F_{\psi(C^2 \times U)}[p(z) + \alpha \frac{zp'(z)}{p(z)}] \leq F_{\psi(C^2 \times U)}[q(z) + \alpha \frac{q'(z)}{q(z)}] = F_{h(U)}h(z), \]
implies
\[ F_{p(U)}p(z) \leq F_{q(U)}q(z) \text{ i.e. } p(z) <_{F} q(z), z \in U, \]
and \(q\) is the best dominant.

\Proof Define the function \(\theta\) and \(\phi\) by \(\theta(w) = w, \phi(w) = \frac{a}{w}, D = \{w : w \neq 0\}\)
in Proposition 2.1. Then the functions
\[ Q(z) = zq'(z)\phi[q(z)] = \alpha \frac{zq'(z)}{q(z)}, \]
Proposition 2.4. Let $q(z)$ is starlike, we obtain that $Q$ is starlike in $U$ and $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for all $z \in U$. It follows Proposition 2.1 and

$$F_{\psi(C^{2} \times U)}[p(z) + \alpha \frac{zp'(z)}{p(z)}] \leq F_{h(U)}h(z),$$

$$F_{p(U)}p(z) \leq F_{q(U)}q(z) \text{ i.e. } p(z) < F_{q(U)}q(z), z \in U,$$

and $q$ is the best dominant. \qed

Proposition 2.3. Let $q \in H[p, p]$ be univalent, $q(z) \neq 0$ and satisfies the conditions:

(i) $\frac{zq'(z)}{q(z)}$ is starlike,

(ii) $Re\left(\frac{q(z)}{\alpha} + 1 + \frac{zq''(z)}{q(z)}\right) > 0$

for $\alpha \neq 0$ and for all $z \in U$. For $f \in A_p$ with

$$J(\alpha, f; z) = (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f(z)}), z \in U$$

and $\psi : C^2 \times U \to C$,

$$\psi(q(z), zq'(z)) = q(z) + \alpha \frac{zq'(z)}{q(z)}, \text{ then}$$

$$F_{\psi(C^{2} \times U)}\left(\frac{zf'(z)}{f(z)}\right) \leq F_{q(U)}q(z)$$

and $q$ is the best dominant.

Proof. Let us put $p(z) = \frac{zf'(z)}{f(z)}, z \in U$, where $p(0) = 0$.

Then we obtain that

$$p(z) + \alpha \frac{zp'(z)}{p(z)} = J(\alpha, f; z).$$

Using Proposition 2.1, we have

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), z \in U,$$

and $q$ is the best dominant. \qed

Proposition 2.4. Let $q \in H[1, 1]$ be univalent and satisfies the following conditions:

(i) $q(z)$ is convex,

(ii) $Re\left(\frac{1}{\alpha} + \rho + \frac{zq''(z)}{q(z)}\right) > 0$ $\rho \in N = \{1, 2, 3, ..\}$

for $\alpha \neq 0$ and for all $z \in U$. For $p \in H[1, 1]$ in $U$ and

$\psi : C^2 \times U \to C$,

$\psi(p(z), zp'(z)) = (1 - \alpha + \alpha \rho)p(z) + \alpha zp'(z)$ is analytic in $U$, then

$$F_{\psi(C^{2} \times U)}[(1 - \alpha + \alpha \rho)p(z) + \alpha zp'(z)] \leq$$

$$F_{\psi(C^{2} \times U)}[(1 - \alpha + \alpha \rho)q(z) + \alpha zq'(z)] = F_{h(U)}h(z),$$

implies $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, and $q$ is the best dominant.
Proof. For $\alpha \neq 0$ real number, we define the functions $\theta$ and $\phi$ by

$$\theta(w) = (1 - \alpha + \alpha \rho)w, \phi(w) = \alpha,$$ $D = \{w : w \neq 0\}$ in Proposition 2.1.

Then we have

(i) $Q(z) = zq'(z)\phi[q(z)] = \alpha zq'(z),$

(ii) $h(z) = \theta[q(z) + Q(z)] = (1 - \alpha + \mu \rho)\phi(q(z) + \alpha zq'(z)).$

By the (i) and (ii), we obtained that $Q$ is starlike in $U$ and $Re(\frac{zq'(z)}{Q(z)}) > 0$
for all $z \in U$. Since it satisfies preconditions of Proposition 2.1, it follows Proposition 2.1,

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), z \in U,$$

and $q$ is the best dominant. \hfill \Box

**Theorem 2.1.** Let $q \in \mathcal{H}[1,1]$ be univalent and satisfies the following conditions:

(i) $q(z)$ is convex,

(ii) $Re\left[\left(\frac{1}{\alpha} + \rho\right) + \frac{zq''(z)}{q'(z)}\right] > 0$ ($\rho \in \mathbb{N} = \{1, 2, 3, \ldots\}$)

for $\alpha \neq 0$ and for all $z \in U$. For $f \in A_p$ with

$$J(\alpha, f; z) = (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)}), z \in U$$

and if $\psi : \mathbb{C}^2 \times U \to \mathbb{C},$

$$\psi(q(z), zq'(z)) = (1 - \alpha + \alpha \rho)q(z) + \mu zq'(z),$$

then

$$F_{\psi(\mathbb{C}^2 \times U)}(\frac{f(z)}{zp}) \leq F_{q(U)}q(z), z \in U$$

and $q$ is the best dominant.

Proof. Let us put $p(z) = \frac{f(z)}{zp}$, where $p(0) = 1$. Then we have

$$1 - \alpha + \alpha \rho)p(z) + \alpha zp'(z) = J_p(\alpha, f; z).$$

From the Proposition 2.4, we have

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), z \in U$$

and $q$ is the best dominant. \hfill \Box

**Corollary 2.1.** Let $q \in \mathcal{H}[1,1]$ be univalent and satisfies the following conditions:

(i) $q(z)$ is convex,

(ii) $Re\left[\left(\frac{1}{\alpha} + 1\right) + \frac{zq''(z)}{q'(z)}\right] > 0$ ($\rho \in \mathbb{N} = \{1, 2, 3, \ldots\}$)

for $\alpha \neq 0$ and for all $z \in U$. For $p \in \mathcal{H}[1,1]$ in $U$,

$$\psi : \mathbb{C}^2 \times U \to \mathbb{C}$$

$$\psi(p(z), zp'(z)) = p(z) + \alpha zp'(z),$$

then

$$F_{\psi(\mathbb{C}^2 \times U)}p(z) \leq F_{q(U)}q(z), z \in U,$$ and $q$ is the best dominant.
Corollary 2.2. Let $q \in \mathcal{H}[1, 1]$ be univalent, $q(z)$ is convex for all $z \in U$. For $p \in \mathcal{H}[1, 1]$ in $U$ if
\[
\psi : \mathbb{C}^2 \times U \to \mathbb{C}, \; \psi(p(z), zp'(z)) = p(z) + zp'(z), \text{ then}
\]
\[
F_{\psi(\mathbb{C}^2 \times U)} p(z) \leq F_{\psi(\mathbb{C}^2 \times U)} q(z), \; z \in U,
\]
and $q$ is the best dominant.

Corollary 2.3. Let $q \in \mathcal{H}[1, 1]$ be univalent, $q(z)$ is convex for all $z \in U$. For $p \in \mathcal{H}[1, 1]$ in $U$ if
\[
\psi : \mathbb{C}^2 \times U \to \mathbb{C}, \; \psi(p(z), zp'(z)) = \rho p(z) + zp'(z), \; (\rho \in \mathbb{N} = \{1, 2, 3, ..\}),
\]
then
\[
F_{\psi(\mathbb{C}^2 \times U)} p(z) \leq F_{\psi(\mathbb{C}^2 \times U)} q(z), \; z \in U,
\]
and $q$ is the best dominant.

REFERENCES


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