$p^*$-Closure Operator and $p^*$-Regularity in Fuzzy Setting

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Abstract. In this paper a new type of fuzzy regularity, viz. fuzzy $p^*$-regularity has been introduced and studied by a newly defined closure operator, viz., fuzzy $p^*$-closure operator. Also we have found the mutual relationship of this closure operator among other closure operators defined earlier. In $p^*$-regular space, $p^*$-closure operator is an idempotent operator. In the last section, $p^*$-closure operator has been characterized via $p^*$-convergence of a fuzzy net.

1. Introduction

Throughout the paper, by $(X, \tau)$ or simply by $X$ we mean a fuzzy topological space (fts, for short) in the sense of Chang [3]. A fuzzy set [7] $A$ is a mapping from a nonempty set $X$ into a closed interval $I = [0,1]$. The support [6] of a fuzzy set $A$ in $X$ will be denoted by $\text{supp}A = \{x \in X : A(x) \neq 0\}$. A fuzzy point [6] with the singleton support $x \in X$ and the value $t$ $(0 < t \leq 1)$ at $x$ will be denoted by $x_t$. $0_X$ and $1_X$ are the constant fuzzy sets taking values 0 and 1 in $X$ respectively. The complement [7] of a fuzzy set $A$ in $X$ will be denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for all $x \in X$. For two fuzzy sets $A$ and $B$ in $X$, we write $A \leq B$ if and only if $A(x) \leq B(x)$, for each $x \in X$, and $AqB$ means $A$ is quasi-coincident (q-coincident, for short) with $B$ [6] if $A(x) + B(x) > 1$, for some $x \in X$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not\q B$ respectively. cl$A$ and int$A$ of a fuzzy set $A$ in $X$ respectively stand for the fuzzy closure [3] and fuzzy interior [3] of $A$ in $X$. A fuzzy set $A$ in $X$ is called fuzzy $\alpha$-open [2] if $A \leq \text{intcl} \text{int}A$. The complement of a fuzzy $\alpha$-open set is called a fuzzy $\alpha$-closed [2] set. The smallest fuzzy $\alpha$-closed set containing a fuzzy set $A$ is called fuzzy $\alpha$-closure

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of $A$ and is denoted by $\alpha clA$ [2], i.e.,

$$\alpha clA = \bigwedge \{ U : A \leq U \text{ and } U \text{ is fuzzy } \alpha \text{-closed} \}$$

A fuzzy set $A$ in $X$ is fuzzy $\alpha$-closed if $A = \alpha clA$ [2]. A fuzzy set $B$ is called a quasi neighbourhood (q-nbd, for short) of a fuzzy set $A$ in an fts $X$ if there is a fuzzy open set $U$ in $X$ such that $A q U \leq B$. If, in addition, $B$ is fuzzy open (resp., $\alpha$-open) then $B$ is called a fuzzy open (resp., $\alpha$-open) q-nbd of $A$. In particular, a fuzzy set $B$ in $X$ is a fuzzy open (resp., $\alpha$-open) q-nbd of a fuzzy point $x_t$ in $X$ if $x_t q U \leq B$, for some fuzzy open (resp., $\alpha$-open) set $U$ in $X$.

2. Fuzzy $p^\ast$-Closure Operator: Some Properties

In this section fuzzy $p^\ast$-closure operator has been introduced and studied. Let us recall a definition from [4] for ready reference.

**Definition 2.1** ([4]). A fuzzy set $A$ in an fts $(X, \tau)$ is called fuzzy preopen if $A \leq \text{int cl}A$. The complement of a fuzzy preopen set is called a fuzzy preclosed set.

The union of all fuzzy preopen sets contained in a fuzzy set $A$ is called fuzzy preinterior of $A$, to be denoted by $pintA$.

The intersection of all fuzzy preclosed sets containing a fuzzy set $A$ is called fuzzy preclosure of $A$, to be denoted by $pclA$.

**Definition 2.2.** A fuzzy preopen set $A$ in an fts $(X, \tau)$ is called a fuzzy pre-q-nbd of a fuzzy point $x_t$, if $x_t q A$.

**Lemma 2.1.** For a fuzzy point $x_t$ and a fuzzy set $A$ in an fts $(X, \tau)$, $x_t \in pclA$ if and only if every fuzzy pre-q-nbd $U$ of $x_t$, $U q A$.

**Proof.** Let $x_t \in pclA$ and $U$ be any fuzzy pre-q-nbd of $x_t$. Then $U(x) + t > 1 \Rightarrow t > 1 - U(x) \Rightarrow x_t \notin 1_X \setminus U$ which is fuzzy preclosed in $X$ and hence by Definition 2.1, $A \nsubseteq 1_X \setminus U \Rightarrow$ there exists $y \in X$ such that $A(y) > (1_X \setminus U)(y) \Rightarrow A(y) + U(y) > 1 \Rightarrow A q U$.

Conversely, let for any fuzzy pre-q-nbd $U$ of $x_t$, $U q A$. Let $V$ be any fuzzy preclosed set containing $A$, i.e., $A \leq V$ ... (1). We have to show that $x_t \in V$. If possible, let $x_t \notin V$. Then $V(x) < t \Rightarrow 1 - V(x) > 1 - t \Rightarrow x_t q (1_X \setminus V)$. By assumption $(1_X \setminus V) q A \Rightarrow A > V$, contradicts (1). \hfill $\square$

**Lemma 2.2.** For any two fuzzy preopen sets $A$ and $B$ in an fts $X$, $A \nsubseteq B \Rightarrow pclA \nsubseteq B$ and $A \nsubseteq pclB$.

**Proof.** If possible, let $pclA q B$. Then there exists $x \in X$ such that $pclA(x) + B(x) > 1$. Let $pclA(x) = t$. Then $B(x) + t > 1 \Rightarrow x_t q B$ and $x_t \in pclA$. By Lemma 2.1, $B q A$, a contradiction.

Similarly, we can prove that $A \nsubseteq pclB$. \hfill $\square$
Definition 2.3. A fuzzy point $x_t$ in an fts $X$ is called fuzzy $p^*$-cluster point of a fuzzy set $A$ in $X$ if $pclUqA$ for every fuzzy pre-q-nbd $U$ of $x_t$.


Note 2.1. It is clear from Definition 2.1 and Definition 2.3 that $pclA \leq [A]_p$, for any fuzzy set $A$ in an fts $X$. The converse is not true, in general, as seen from the following example.

Example 2.1. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, B\}$ where $B(a) = 0.7$, $B(b) = 0.5$. Then $(X, \tau)$ is an fts. The collection of all fuzzy preopen sets in $(X, \tau)$ is of the form $\{0_X, 1_X, B, U\}$ where $U \not\subseteq 1_X \setminus B$ and that of fuzzy preclosed sets is $\{0_X, 1_X, 1_X \setminus B, 1_X \setminus U\}$ where $1_X \setminus U \not\supseteq B$. Consider the fuzzy point $a_{0.2}$ and the fuzzy set $V$ defined by $V(a) = V(b) = 0.1$. Then $a_{0.2} \notin pclV$ but $a_{0.2} \in [V]_p$. Indeed, $C(a) = 0.81, C(b) = 0.5$ is a fuzzy pre-q-nbd of $a_{0.2}$ but $C \not\mathcal{V}$. Although $pclC = 1_XqV$.

The following theorem shows that under which condition, the two closure operators $pcl$ and $p^*$ coincide.

Theorem 2.1. For a fuzzy preopen set $A$ in an fts $(X, \tau)$, $[A]_p = pclA$.

Proof. By Note 2.1, it suffices to show that $[A]_p \leq pclA$, for any fuzzy preopen set $A$ in $X$.

Let $x_t$ be a fuzzy point in $X$ such that $x_t \notin pclA$. Then there exists a fuzzy pre-q-nbd $V$ of $x_t$ such that $V \not\mathcal{V}A$. Then $V(y) + A(y) \leq 1$, for all $y \in X \Rightarrow V(y) \leq 1 - A(y)$, for all $y \in X \Rightarrow pclV \leq pcl(1_X \setminus A) = 1_X \setminus A$ (since $1_X \setminus A$ is fuzzy preclosed in $X$). Thus $pclV \not\mathcal{V}A$ and consequently, $x_t \notin [A]_p$. Hence $[A]_p \leq pclA$ for a fuzzy preopen set $A$ in $X$. □

We now characterize fuzzy $p^*$-closure operator of a fuzzy set $A$ in an fts $X$.

Theorem 2.2. For any fuzzy set $A$ in an fts $(X, \tau)$, $[A]_p = \cap\{[U]_p: U \text{ is fuzzy preopen in } X \text{ and } A \subseteq U\}$.

Proof. Clearly, L.H.S. $\leq$ R.H.S.

If possible, let $x_t \in$ R.H.S. but $x_t \notin$ L.H.S. Then there exists a fuzzy pre-q-nbd $V$ of $x_t$ such that $pclV \not\mathcal{V}A$ and so $A \leq 1_X \setminus pclV$ and $1_X \setminus pclV$ being fuzzy preopen set in $X$ containing $A$, by our assumption, $x_t \in [1_X \setminus pclV]_p$. But $pclV \not\mathcal{V}(1_X \setminus pclV)$ and so $x_t \notin [1_X \setminus pclV]_p$, a contradiction. This completes the proof. □

Remark 2.1. By Theorem 2.1 and Theorem 2.2, we can conclude that $[A]_p$ is fuzzy preclosed in $X$ for a fuzzy set $A$ in $X$.

Theorem 2.3. In an fts $(X, \tau)$, the following hold:
(a) the fuzzy sets $0_X$ and $1_X$ are fuzzy $p^*$-closed sets in $X$,
(b) for two fuzzy sets $A$ and $B$ in $X$, if $A \leq B$, then $[A]_p \leq [B]_p$,
(c) the intersection of any two fuzzy $p^*$-closed sets in $X$ is fuzzy $p^*$-closed in $X$.

Proof. (a) and (b) are obvious.
(c) Let $A$ and $B$ be any two fuzzy $p^*$-closed sets in $X$. Then $A = [A]_p$ and $B = [B]_p$. Now $A \wedge B \leq A$, $A \wedge B \leq B$. Then by (b), $[A \wedge B]_p \leq [A]_p$ and $[A \wedge B]_p \leq [B]_p$. Therefore, $[A \wedge B]_p \leq [A]_p \wedge [B]_p = A \wedge B$.

Conversely, let $x_t \in A \wedge B$. Then $x_t \in [A]_p$ and $x_t \in [B]_p$. Then $A(x) \geq t, B(x) \geq t$, i.e., $(A \wedge B)(x) = \min\{A(x), B(x)\} \geq t$. Now for any fuzzy pre-q-nbd $V$ of $x_t$, $pclVqA, pclVqB$. Then $V(x) + t > 1$. Therefore, $pclV(x) + (A \wedge B)(x) > 1 - t + t = 1$. Therefore, $pclVq(A \wedge B)$ for any fuzzy pre-q-nbd $V$ of $x_t$ and hence $x_t \in [A \wedge B]_p$. Consequently, $[A]_p \wedge [B]_p \leq [A \wedge B]_p$. □

Remark 2.2. In fact, the intersection of any collection of fuzzy $p^*$-closed sets is fuzzy $p^*$-closed. But the union of two fuzzy $p^*$-closed sets may not be fuzzy $p^*$-closed is clear from the following example.

Example 2.2. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.7$. Then $(X, \tau)$ is an fts. The collection of all fuzzy preopen sets in $(X, \tau)$ is $\{0_X, 1_X, A, U\}$ where $U \not\leq 1_X \wedge A$. Then the collection of all fuzzy preclosed sets is $\{0_X, 1_X, 1_X \wedge A, 1_X \wedge U\}$ where $1_X \wedge U \not\geq A$. Let $C$ and $D$ be two fuzzy sets given by $C(a) = 0.5, C(b) = 0.6, D(a) = 0.2, D(b) = 0.7$. Then $(C \vee D)(a) = 0.5, (C \vee D)(b) = 0.7$. Now $a_{0.6} \notin [C]_p$ as $a_{0.6}qU$ where $U(a) = 0.41, U(b) = 0.31$, but $pclU = U \not\bigvee C$. Again $a_{0.6} \notin [D]_p$ as $a_{0.6}qV$ where $V(a) = 0.7, V(b) = 0.2$, but $pclV = V \not\bigvee D$.

But for any fuzzy pre-q-nbd of $a_{0.6}$ is of the form $U$ where $U \not\leq 1_X \wedge A$. Then $pclU = Uq(C \vee D)$ and consequently, $a_{0.6} \in [C \vee D]_p$. Therefore, $[C]_p \vee [D]_p = [C \vee D]_p$. Also $(C \vee D)(a) = 0.5 \not\geq 0.6$ and so $a_{0.6} \notin C \vee D$.

Note 2.2. It is clear from Remark 2.2 that fuzzy $p^*$-open sets in an fts $(X, \tau)$ may not form a base for a fuzzy topology.

Result 2.1. We conclude that $x_t \in [y_t]_p$ does not imply $y_{t'} \in [x_t]_p$ where $x_t, y_{t'}$ (0 < $t, t'$ < 1) are fuzzy points in $X$ as shown from the following example.

Example 2.3. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0, B(a) = 0.7, B(b) = 0$. Then $(X, \tau)$ is an fts. The collection of all fuzzy preopen sets in $X$ is $\{0_X, 1_X, A, B, U, V\}$ where $0.3 < U(a) \leq 0.5, U(b) = 0$ and $V(a) = 0.5, 0 \leq V(b) \leq 1$. Then the collection of all fuzzy preclosed sets is $\{0_X, 1_X, 1_X \wedge A, 1_X \wedge B, 1_X \wedge U, 1_X \wedge V\}$ where $0.5 \leq 1 - U(a) < 0.7, U(b) = 1$ and $0 \leq 1 - V(a) < 0.5, 0 \leq 1 - V(b) \leq 1$. Consider the fuzzy points $a_{0.6}$ and $b_{0.1}$. We claim that $b_{0.1} \in [a_{0.6}]_p$, but $a_{0.6} \notin [b_{0.1}]_p$. Indeed, any fuzzy pre-q-nbd of $b_{0.1}$ is of the form $V$ where $V(a) > 0.5, V(b) > 0.9$ and $pclV = W$. 
where \( W(a) > 0.5, W(b) = 1 \) and \( Wqa_{0.6} \). But \( D(a) = 0.41, D(b) = 0 \) is a fuzzy pre-q-nbd of \( a_{0.6} \) and \( pclD = D \not\subset b_{0.1} \).

3. \( p^* \)-Closure Operator: Mutual Relationship with Other Closure Operators

In this section we have established some mutual relationship of \( p^* \)-closure operator with other closure operators, viz., \( \alpha^* \)-closure operator, \( \theta \)-closure operator.

First We recall some definitions for ready references.

**Definition 3.1 ([5])**. Let \( A \) be a fuzzy set and \( x_t \), a fuzzy point in an fts \( X \). \( x_t \) is called a fuzzy \( \theta \)-cluster point of \( A \) if every closure of every fuzzy open q-nbd of \( x_t \) is \( q \)-coincident with \( A \).

The union of all fuzzy \( \theta \)-cluster points of \( A \) is called fuzzy \( \theta \)-closure of \( A \), to be denoted by \([A]_\theta \). \( A \) is called fuzzy \( \theta \)-closed if \( A = [A]_\theta \) and the complement of a fuzzy \( \theta \)-closed set is called fuzzy \( \theta \)-open.

**Definition 3.2 ([1])**. A fuzzy point \( x_t \) in an fts \( X \) is called a fuzzy \( \alpha^* \)-cluster point of a fuzzy set \( A \) in \( X \) if \( aclUqA \) for every fuzzy \( \alpha \)-open q-nbd \( U \) of \( x_t \).

The union of all fuzzy \( \alpha^* \)-cluster points of \( A \) is called fuzzy \( \alpha^* \)-closure of \( A \), to be denoted by \([A]_{\alpha^*} \). A fuzzy set \( A \) is called fuzzy \( \alpha^* \)-closed if \( A = [A]_{\alpha^*} \) and the complement of fuzzy \( \alpha^* \)-closed set is called fuzzy \( \alpha^* \)-open.

**Result 3.1.** \([A]_p \leq [A]_\theta \), for any fuzzy set \( A \) in an fts \( X \).

*Proof.* Let \( x_t \in [A]_p \). Let \( V \) be any fuzzy open q-nbd of \( x_t \). Then \( V \) is fuzzy pre-q-nbd of \( x_t \) also. As \( x_t \in [A]_p \), \( pclVqA \Rightarrow clVqA \Rightarrow x_t \in [A]_\theta \). \( \square \)

**Remark 3.1.** It is clear from the following example that \([A]_p \neq [A]_\theta \), for any fuzzy set \( A \) in an fts \( X \), in general.

**Example 3.1.** Consider Example 2.1. Consider the fuzzy point \( a_{0.51} \) and a fuzzy set \( C \) given by \( C(a) = C(b) = 0.1 \). Then \( U(a) = 0.5, U(b) = 0 \) being a fuzzy pre-q-nbd of \( a_{0.51} \), \( pclU = U \not\subset C \) and so \( a_{0.51} \notin [C]_p \). But other than \( 1_X \), \( B \) is the only fuzzy open q-nbd of \( a_{0.51} \) and \( clB = 1_X qC \). Therefore, \( a_{0.51} \in [C]_\theta \).

**Result 3.2.** \([A]_p \leq [A]_{\alpha^*} \), for any fuzzy set \( A \) in an fts \( X \).

*Proof.* Let \( x_t \in [A]_p \). Let \( U \) be a fuzzy \( \alpha \)-open q-nbd of \( x_t \). Then \( U \) is a fuzzy preopen set and hence \( pclUqA \Rightarrow aclUqA \Rightarrow x_t \in [A]_{\alpha^*} \). \( \square \)

**Remark 3.2.** It is clear from the following example that \([A]_p \neq [A]_{\alpha^*} \), for any fuzzy set \( A \) in an fts \( X \), in general.

**Example 3.2.** Let \( X = \{a, b\}, \tau = \{0_X, 1_X, A, B\} \) where \( A(a) = 0.5, A(b) = 0.4, B(a) = 0.7, B(b) = 0.5 \). Then \( (X, \tau) \) is an fts. The collection of all fuzzy \( \alpha \)-open sets is \( \{0_X, 1_X, A, B, V\} \) where \( V \geq B \) and that of fuzzy preopen sets is \( \{0_X, 1_X, A, B, U, V_1, W\} \) where \( U \leq A, U \leq 1_X \setminus B, V_1 > 1_X \setminus A, W \geq B \).
Consider the fuzzy point $b_{0.71}$ and the fuzzy set $D$, defined by $D(a) = D(b) = 0.6$. Then $U_1(a) = 0.4, U_1(b) = 0.3$ is a fuzzy preopen set such that $b_{0.71} \notin U_1$. But $pclU_1 = U_1 \not\equiv D$ and so $b_{0.71} \notin [D]_p$. All fuzzy $\alpha$-open q-nbds of $b_{0.71}$ are $1_X, A, B, V$ where $V \geq B$. $\alphacl A = (1_X \setminus A)qD, \alphacl B = \alphacl V = \alphacl 1_X = 1_XqD$ and so $b_{0.71} \notin [D]_p$.

**Remark 3.3.** The following two examples show that fuzzy $p^*$-closure operator and fuzzy closure operator are independent notions.

**Example 3.3.** Let $X = \{a, b\}, \tau = \{0_X, 1_X, B\}$ where $B(a) = 0.7, B(b) = 0.5$. Then $(X, \tau)$ is an fts. Consider the fuzzy point $a_{0.51}$ and the fuzzy set $C$ given by $C(a) = C(b) = 0.4$. Then $a_{0.51} \notin [C]_p$ as $U$ defined by $U(a) = 0.5, U(b) = 0$, being a fuzzy pre-q-nbd of $a_{0.51}, pclU = U \not\equiv qC$. But other than $1_X, B$ is the only fuzzy open q-nbd of $a_{0.51}$ such that $BqC$. Consequently, $a_{0.51} \notin ClC$.

**Example 3.4.** Let $X = \{a\}, \tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.4, B(a) = 0.7$. Then $(X, \tau)$ is an fts. Then the collection of all fuzzy preopen sets is $\{0_X, 1_X, U, V\}$ where $U \leq A, V \geq B$. Consider the fuzzy point $a_{0.4}$ and the fuzzy set $C$ given by $C(a) = 0.3$. Then $B$ is a fuzzy open q-nbd of $a_{0.4}$, but $B \not\equiv qC$ and so $a_{0.4} \notin clC$. But any fuzzy pre-q-nbd of $a_{0.4}$ is of the form $V$ and $pclV = 1_XqC$ and so $a_{0.4} \in [C]_p$.

4. **Fuzzy $p^*$-Regular Space: Some Characterizations**

In this section a new type of fuzzy regularity has been introduced and studied and shown that in this space $p^*$-closure operator and $pcl$ operator coincide.

**Definition 4.1.** An fts $(X, \tau)$ is said to be fuzzy $p^*$-regular if for each fuzzy point $x_t$ and each fuzzy pre-q-nbd $U$ of $x_t$, there exists a fuzzy preopen set $V$ in $X$ such that $x_tqV \leq pclV \leq U$.

**Theorem 4.1.** For an fts $(X, \tau)$, the following conditions are equivalent:

(a) $X$ is fuzzy $p^*$-regular space.
(b) For any fuzzy set $A$ in $X$, $[A]_p = pclA$.
(c) For each fuzzy point $x_t$ and each fuzzy preclosed set $F$ with $x_t \notin F$, there exists a fuzzy preopen set $U$ such that $x_t \notin pclU$ and $F \leq U$.
(d) For each fuzzy point $x_t$ and each fuzzy preclosed set $F$ such that $x_t \notin F$, there exist fuzzy preopen sets $U$ and $V$ in $X$ such that $x_tqU, F \leq V$ and $U \not\equiv V$.
(e) For any fuzzy set $A$ and any fuzzy preclosed set $F$ with $A \not\subseteq F$, there exist fuzzy preopen sets $U$ and $V$ such that $AqU, F \leq V$ and $U \not\equiv qV$.
(f) For any fuzzy set $A$ and any fuzzy preopen set $U$ with $AqU$, there exists a fuzzy preopen set $V$ such that $AqV \leq pclV \leq U$.

**Proof.** (a) ⇒ (b): By Note 2.1, it suffices to show that $[A]_p \leq pclA$, for any fuzzy set $A$ in $X$. 
Let \( x_t \in [A]_p \) and \( V \) be any fuzzy pre-q-nbd of \( x_t \). By (a), there exists a fuzzy preopen set \( W \) such that \( x_t q W \leq pcl W \leq V \). Since \( x_t \in [A]_p \), \( pcl W q A \) and so \( V q A \). Consequently, \( x_t \in pcl A \Rightarrow [A]_p \leq pcl A \).

(b) \( \Rightarrow \) (a): Let \( x_t \) be a fuzzy point in \( X \) and \( U \) be any fuzzy pre-q-nbd of \( x_t \). Then \( U(x) + t > 1 \Rightarrow x_t \notin (1_X \setminus U) = pcl (1_X \setminus U) = [1_X \setminus U]_p \) (by (b)). Then there exists a fuzzy pre-q-nbd \( V \) of \( x_t \) such that \( pcl V \not\emptyset (1_X \setminus U) \Rightarrow pcl V \leq U \). Then \( x_t q V \leq pcl V \leq U \Rightarrow X \) is fuzzy \( p^* \)-regular.

(a) \( \Rightarrow \) (c): Let \( x_t \) be a fuzzy point in \( X \) and \( F \), a fuzzy preclosed set in \( X \) with \( x_t \notin F \). Then \( F(x) < t \Rightarrow 1 - F(x) + t > 1 \Rightarrow x_t q (1_X \setminus F) \). By (a), there exists a fuzzy preopen set \( W \) such that \( x_t q W \leq pcl W \leq 1_X \setminus F \). Therefore, \( F \leq 1_X \setminus pcl W = U \) (say) which is fuzzy preopen. Now \( x_t q W \Rightarrow x_t q pint W \leq W \leq pint (pcl W) \Rightarrow x_t q pint (pcl W) \Rightarrow (pint (pcl W))(x) + t > 1 \Rightarrow 1 - (pint (pcl W))(x) < t \Rightarrow x_t \notin 1_X \setminus (pint (pcl W)) \Rightarrow x_t \notin pcl (1_X \setminus pcl W) \Rightarrow x_t \notin pcl U \).

(c) \( \Rightarrow \) (d): Let \( x_t \) be a fuzzy point in \( X \) and \( F \), a fuzzy preclosed set in \( X \) with \( x_t \notin F \). By (c), there exists a fuzzy preopen set \( U \) such that \( x_t \notin pcl U \) and \( F \leq U \). Now \( x_t \notin pcl U \Rightarrow \) there exists a fuzzy pre-q-nbd \( W \) of \( x_t \) such that \( W \not\emptyset \).

(d) \( \Rightarrow \) (e): Let \( A \) be any fuzzy set and \( F \), any fuzzy preclosed set in \( X \) with \( A \leq F \). Then there exists \( x \in X \) such that \( A(x) > F(x) \). Let \( A(x) = t \). Then \( x_t \notin F \). By (d), there exist fuzzy preopen sets \( U \) and \( V \) such that \( x_t q U, F \leq V \) and \( U \not\emptyset \). Again, \( U(x) + A(x) = U(x) + t > 1 \Rightarrow Aq U \).

(e) \( \Rightarrow \) (f): Let \( A \) be any fuzzy set and \( U \), any fuzzy preopen set in \( X \) with \( Aq U \). Then \( A \leq 1_X \setminus U \) which is fuzzy preclosed. By (e), there exist fuzzy preopen sets \( V \) and \( W \) such that \( Aq V, 1_X \setminus U \leq W \) and \( V \not\emptyset \). Then by Lemma 2.2, \( pcl V \not\emptyset \). Thus \( Aq V \leq pcl V \leq 1_X \setminus W \leq U \).

(f) \( \Rightarrow \) (a): Obvious. \( \square \)

**Corollary 4.1.** An fts \( (X, \tau) \) is fuzzy \( p^* \)-regular if and only if every fuzzy preclosed set in \( X \) is fuzzy \( p^* \)-closed in \( X \).

**Proof.** Let \((X, \tau)\) be fuzzy \( p^* \)-regular space and \( A \), a fuzzy preclosed set in \( X \). Then by Theorem 4.1 (a) \( \Rightarrow \) (b), \( A = pcl A = [A]_p \) and hence \( A \) is fuzzy \( p^* \)-closed in \( X \).

Conversely, let \( A = [A]_p \) for any fuzzy preclosed set in \( X \). Let \( B \) be any fuzzy set in \( X \). Then \( pcl B = [pcl B]_p \). Then \([B]_p \leq [pcl B]_p = pcl B \). Again from Note 2.1, \( pcl B \leq [B]_p \) and so \([B]_p = pcl B \) for any fuzzy set \( B \) in \( X \). Hence by Theorem 4.1 (b) \( \Rightarrow \) (a), \( X \) is fuzzy \( p^* \)-regular space. \( \square \)

**Remark 4.1.** In a fuzzy \( p^* \)-regular space \( (X, \tau) \), \([A]_p \) is fuzzy \( p^* \)-regular space.

**Proof.** By Theorem 4.1 (a) \( \Rightarrow \) (b), \([A]_p = [pcl A]_p = pcl (pcl A) = pcl A = [A]_p \) (by Theorem 4.1 (a) \( \Rightarrow \) (b)). \( \square \)
5. CHARACTERIZATIONS OF FUZZY $p^*$-CLOSURE OPERATOR VIA FUZZY NET

In this section fuzzy $p^*$-closure operator of a fuzzy set is characterized in terms of fuzzy $p^*$-cluster point of a fuzzy net and its fuzzy $p^*$-convergence.

**Definition 5.1.** A fuzzy point $x_t$ in an fts $(X, \tau)$ is called a fuzzy $p^*$-cluster point of a fuzzy net $\{S_n : n \in (D, \geq)\}$ if for every fuzzy pre-q-nbd $U$ of $x_t$ and for any $n \in D$, there exists $m \in D$ with $m \geq n$ such that $S_m qpclU$.

**Definition 5.2.** A fuzzy net $\{S_n : n \in (D, \geq)\}$ in an fts $(X, \tau)$ is said to $p^*$-converge to a fuzzy point $x_t$ if for any fuzzy pre-q-nbd $U$ of $x_t$, there exists $m \in D$ such that $S_n qpclU$ for all $n \geq m$ ($n \in D$). This is denoted by $\xrightarrow{S_n p^*} x_t$.

**Theorem 5.1.** A fuzzy point $x_t$ is a fuzzy $p^*$-cluster point of a fuzzy net $\{S_n : n \in (D, \geq)\}$ in an fts $(X, \tau)$ if and only if there exists a fuzzy subnet of $\{S_n : n \in (D, \geq)\}$ which $p^*$-converges to $x_t$.

**Proof.** Let $x_t$ be a fuzzy $p^*$-cluster point of the fuzzy net $\{S_n : n \in (D, \geq)\}$. Let $p(Q_{x_t})$ denote the set of fuzzy preclosures of all fuzzy pre-q-nbds of $x_t$. Then for any $A \in p(Q_{x_t})$, there exists $n \in D$ such that $S_n qA$. Let $E$ denote the set of all ordered pairs $(n, A)$ such that $n \in D$, $A \in p(Q_{x_t})$ and $S_n qA$. Then $(E, \gg)$ is a directed set, where $(m, A) \gg (n, B)$ if and only if $m \geq n$ and $A \leq B$. Then $T : (E, \gg) \rightarrow (X, \tau)$ given by $T(m, A) = S_m$ is clearly a fuzzy subnet of $\{S_n : n \in (D, \geq)\}$.

We claim that $\xrightarrow{T p^*} x_t$. Let $V$ be any fuzzy pre-q-nbd of $x_t$. Then there exists $n \in D$ such that $(n, pclV) \in E$ and so $S_n qpclV$. Now for any $(m, A) \gg (n, pclV)$, $T(m, A) = S_m qA \leq pclV \Rightarrow T(m, A) qpclV$. Consequently, $\xrightarrow{T p^*} x_t$.

Conversely, if $x_t$ is not a fuzzy $p^*$-cluster point of the fuzzy net $\{S_n : n \in (D, \geq)\}$, then there exists a fuzzy pre-q-nbd $U$ of $x_t$ and an $n \in D$ such that $S_m \not\in qpclU$, for all $m \geq n$. Then clearly, no fuzzy subnet of the net $\{S_n : n \in (D, \geq)\}$ can $p^*$-converge to $x_t$.

**Theorem 5.2.** Let $A$ be a fuzzy set in an fts $(X, \tau)$. A fuzzy point $x_t \in [A]_p$ if and only if there exists a fuzzy net $\{S_n : n \in (D, \geq)\}$ in $A$, which $p^*$-converges to $x_t$.

**Proof.** Let $x_t \in [A]_p$. Then for any fuzzy pre-q-nbd $U$ of $x_t$, pclU$qA$, i.e., there exists $y^U \in suppA$ and a real number $p_U$ with $0 < p_U \leq A(y^U)$ such that the fuzzy point $y^U_{pq}$ with support $y^U$ and value $p_U$ belong to $A$ and $y^U_{pq} qpclU$. We choose and fix one such $y^U_{pq}$, for each $U$. Let $D$ denote the set of all fuzzy pre-q-nbds of $x_t$. Then $(D, \geq)$ is a directed set under inclusion relation, i.e., $B, C \in D$, $B \geq C$ iff $B \leq C$. Then $\{y^U_{pq} : U \in D\}$ is a fuzzy net in $A$ such that it $p^*$-converges to $x_t$. Indeed, for any fuzzy pre-q-nbd $U$ of $x_t$, if $V \in D$ and $V \geq U$ (i.e., $V \leq U$), then $y^V_{pq} qpclV \leq pclU \Rightarrow y^V_{pq} qpclU$. 

\[ p^* - Closure Operator and p^* - Regularity in Fuzzy Setting \]
Conversely, let \( \{ S_n : n \in (D, \geq) \} \) be a fuzzy net in \( A \) such that \( S_n \overset{p}{\rightarrow} x_t \). Then for any fuzzy pre-q-nbd \( U \) of \( x_t \), there exists \( m \in D \) such that \( n \geq m \Rightarrow S_nqpclU \Rightarrow AqpclU \) (since \( S_n \in A \)). Hence \( x_t \in [A]_p \).

\[ \square \]

**Remark 5.1.** It is clear that an improved version of the converse of the last theorem can be written as “\( x_t \in [A]_p \) if there exists a fuzzy net in \( A \) with \( x_t \) as a fuzzy \( p^* \)-cluster point”.

**References**


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