On a General Class of $q$-Rational Type Operators

NURHAYAT İSPIR

ABSTRACT. In this study, we define a general class of rational type operators based on $q$-calculus and investigate the weighted approximation properties of these operators by using $A$-statistical convergence. We also estimate the rates of $A$-statistical convergence of these operators by modulus of continuity and Petree’s $K$-functional. The operators to be introduced, include some well known $q$-operators so our results are true in a large spectrum of these operators.

1. Introduction


The generalization of the Bernstein type rational operators are introduced by C. Atakut and N. Ispir [16] as follows

\[ L_n(f)(x) = \frac{1}{\psi_n(a_n x)} \sum_{k=0}^{\infty} f \left( \frac{k}{b_n} \right) \frac{\psi_n^{(k)}(0)}{k!} (a_n x)^k, \quad n \in \mathbb{N}, \quad x \geq 0, \]

where $a_n$ and $b_n$ are suitably chosen real numbers, independent of $x$. Here \{\psi_n\} is a sequence of functions $\psi_n : \mathbb{C} \to \mathbb{C}$ satisfying the following conditions:

a) $\psi_n \ (n = 1, 2, \ldots)$ is analytic on a domain $D$ containing the disk $B = \{z \in \mathbb{C} : |z - b| \leq b\}$;

b) $\psi_n(0) = 1$, \quad ($n = 1, 2, \ldots$);

c) For any $x \geq 0$, $\psi_n(x) > 0$ and $\psi_n^{(k)}(0) \geq 0$ for any $n = 1, 2, \ldots$, $k = 1, 2, \ldots$.

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d) For every \( n = 1, 2, \ldots \), it is

\[
\lim_{n} \left( \frac{\psi_{n}(\nu)(a_{n}x)}{n^{\nu}\psi_{n}(a_{n}x)} - 1 \right) = 0; \quad \nu = 1, 2,
\]

where \( a_{n} \to 0 \), as \( n \to \infty \).

In [16] the authors estimated the order of approximation for the operators defined by (1) and proved a Voronovskaja type asymptotic formula and pointwise convergence in simultaneous approximation. In [13, 14, 18] the approximation properties for different variants of the operators (1) were investigated in various function spaces. In [17], the approximation properties of the Kantorovich variant of the operators (1) were given by the aid of \( A \)-statistical convergence. Notice that \( A \)-statistical convergence is stronger than usual convergence.

It is known that the applications of \( q \)-calculus in the area of approximation theory have been an active area of research. In the recent years, the statistical approximation properties of some positive linear operators based on \( q \)-integers have been studied intensively by many authors (e.g.[5, 15, 17, 21]).

The aim of this study is to introduce \( q \)-type generalization of the operators (1) and investigate the \( A \)-statistical approximation properties of the constructed operators in weighted spaces. Using \( A \)-statistical convergence, we obtain weighted Korovkin type theorem and weighted order of approximation by the constructed operators based \( q \)-calculus. Moreover, we estimate the rate of \( A \)-statistical convergence by usual modulus of continuity and by Petree’s \( K \)-functional in the different normed spaces for \( q \)-extension of the operators (1).

Now, let us give a few basic definitions and notations in \( q \)-integers shortly. Details on \( q \)-calculus can be found in [7]. Throughout the present paper, we consider \( q \) as a real number such that \( 0 < q < 1 \), and for each nonnegative integer \( i \), the \( q \)-integer \([i]_q\) is defined by \([i]_q = (1 - q^i) / (1 - q)\), \([0]_q := 0\); and \( q \)-factorial \([i]_q!\) is defined by \([i]_q! = [1]_q [2]_q \ldots [i]_q\), \([0]_q! := 1\).

For fixed \( 0 < q < 1 \), the \( q \)-derivative of a function \( f : \mathbb{R} \to \mathbb{R} \) with respect to \( x \) is defined by \( D_q f (x) = \frac{f(qx) - f(x)}{(q-1)x}, x \neq 0 \) and \( D_q f (0) = \lim_{x \to 0} D_q f (x) \). The chain rule for ordinary derivatives is similar for \( q \)-derivative.

At this point, we recall the \( q \)-Taylor theorem in the following.

**Theorem A** ([7], p. 103.) If a function \( f(x) \) possess convergence series expansion then

\[
f(x) = \sum_{n=0}^{\infty} \frac{(x - a)^n}{[n]_q^n} D_q^n [f(a)]
\]

where \( (x - a)_q^n = \prod_{s=0}^{n-1} (x - q^sa) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] q^{k(k-1)} x^{n-k} (-a)^k \).

Now let us recall some concepts of the \( A \)-statistical convergence. Suppose that \( A \) is non-negative summability matrix and let \( K \) be subset of \( \mathbb{N} \).
Choosing and taking $x^{\infty}_{q}$ in normed spaces: Assume that the concept of $A$-statistical convergence may also be given in normed spaces: Assume $(X, \|\|)$ is a normed space and $u = (u_k)$ is a $X$-valued sequence. Then $(u_k)$ is said to be $A$-statistically convergent to $u_0 \in X$ if, for every $\varepsilon > 0$, $\delta_A \{ k \in \mathbb{N} : \|u_k - u_0\| \geq \varepsilon \} = 0$ [19].

The case in which $A = C_1$, the Cesáro matrix of order one, $A$-statistical convergence reduces to the statistically convergence [9, 19]. Also if $A = I$, the identity matrix, then it reduces to the ordinary convergence. We note that, if $A = (a_{jn})$ is a non-negative regular matrix such that $\lim\sup_{n} \{a_{jn}\} = 0$,

then $A$-statistical convergence is stronger than convergence [19]. It should be noted that the concept of $A$-statistical convergence may also be given in normed spaces: Assume $(X, \|\|)$ is a normed space and $u = (u_k)$ is a $X$-valued sequence. Then $(u_k)$ is said to be $A$-statistically convergent to $u_0 \in X$ if, for every $\varepsilon > 0$, $\delta_A \{ k \in \mathbb{N} : \|u_k - u_0\| \geq \varepsilon \} = 0$ [19].

2. CONSTRUCTION OF OPERATORS AND AUXILIARY RESULTS

Now we would like to introduce $q$-generalization of the operators (1). Let $(\varphi_n)$ be a sequence of real functions on $\mathbb{R}_+$ which are continuously infinitely $q$-differentiable on $\mathbb{R}_+$ satisfying the following conditions

1. $\varphi_n(0) = 1$, for each $n \in \mathbb{N}$
2. $D_q^k \varphi_n(0) \geq 0$, for every $n, k \in \mathbb{N}, x \geq 0$
3. For every $n \in \mathbb{N}$, with $a_n = [n]_q^{\beta-1}, b_n = [n]_q^\beta, q \in (0, 1], 0 < \beta \leq \frac{2}{3}$

$st_A - \lim_{n} \left( \frac{D_q^\nu \varphi_n([n]_q^{\beta-1} x)}{q^{\nu-1} [n]_q^\nu \varphi_n([n]_q^{\beta-1} x)} - 1 \right) = 0; \ \nu = 1, 2.$

(3)

For fixed $x \in \mathbb{R}_+$, taking account to Theorem A we get

$\varphi_n \left( [n]_q^{\beta-1} x \right) = \sum_{k=0}^{\infty} \frac{\left( [n]_q^{\beta-1} x \right)^k}{[k]_q!} D_q^k \varphi_n(0).$

(4)

Using the formula $(a + b)^n_q = \prod_{s=0}^{n-1} (a + q^s b) = \sum_{k=0}^{n} q^{k(k-1)/2} \left( \begin{array}{c} n \\ k \end{array} \right)_q b^k a^{n-k}$

and taking $a = 0, b = [n]_q^{\beta-1} x$ we write $\left( [n]_q^{\beta-1} x \right)_q^k = q^{k(k-1)/2} \left( [n]_q^{\beta-1} x \right)^k.$

Choosing $a_n = [n]_q^{\beta-1}, b_n = [n]_q^\beta, 0 < \beta \leq \frac{2}{3}, n \in \mathbb{N},$ we introduce $q$-generalization of the operator (1) as follows

$L_{n,q}(f)(x) = \frac{1}{\varphi_n([n]_q^{\beta-1} x)} \sum_{k=0}^{\infty} f \left( \frac{[k]_q}{[n]_q^{\beta-1} x} \right) q^{k(k-1)/2} \left( [n]_q^{\beta-1} x \right)^k \frac{[k]_q!}{[k]_q!} D_q^k \varphi_n(0),$

(5)
Lemma 2.1. For all $n \in \mathbb{N}$, $x \geq 0$.

Note that:

- It is easily verified that the operators $L_{n,q}$ are linear positive operators.
- The order of convergence is the best possible estimate for $\beta \in (0, 2/3]$ (see [4]).
- The present condition (3) is weaker than the present one given by (2) for $q = 1$. Indeed, we can construct a sequence such that it is statistically convergent to 1 but not convergent in the ordinary sense. A well known example is defined as; $\alpha_n = \sqrt{n}$ if $n = m^2$ ($m \in \mathbb{N}$), and $\alpha_n = 1$ otherwise. Same result also works to $A$–statistical convergence.

**Lemma 2.1.** For all $n \in \mathbb{N}$, $x \geq 0$ and $0 < q < 1$ we get

(6) $L_{n,q}(e_0)(x) = 1,$

(7) $L_{n,q}(e_1)(x) = \frac{D_q \varphi_n ([n]_q^{\beta-1} x)}{[n]_q \varphi_n ([n]_q^{\beta-1} x)},$

(8) $L_{n,q}(e_2)(x) = \frac{D_q^{2} \varphi_n ([n]_q^{\beta-1} x)}{q [n]_q^{2} \varphi_n ([n]_q^{\beta-1} x)} x^2 + \frac{1}{[n]_q^\beta} \frac{D_q \varphi_n ([n]_q^{\beta-1} x)}{\varphi_n ([n]_q^{\beta-1} x)} x$

where $(e_i)(x) = x^i$, $i = 0, 1, 2$.

**Proof.** From (4) and definition of $L_{n,q}(f)$ it is clear that $L_{n,q}(e_0)(x) = 1$. Considering (4), we can write the $q$-derivative of $\varphi_n$ with respect to $x$ as

(9) $[n]_q^{\beta-1} D_q \varphi_n ([n]_q^{\beta-1} x) = \sum_{k=1}^{\infty} \frac{[k]_q}{[k]_q !} [n]_q^{\beta-1} ([n]_q^{\beta-1} x) q^{k-1} D_q^{k} \varphi_n(0)$

$$= \sum_{k=1}^{\infty} \frac{[k]_q}{[k]_q !} [n]_q^{\beta-1} q^{(k-1)(k-2)/2} ([n]_q^{\beta-1} x) q^{k-1} D_q^{k} \varphi_n(0)$$

where used the equation $(a_n x)_q^k = q^{k(k-1)} (a_n x)^k$. Hence multiplying both sides by $x$ and dividing by $[n]_q^\beta \varphi_n ([n]_q^{\beta-1} x)$ we obtain

(10) $\frac{D_q \varphi_n ([n]_q^{\beta-1} x)}{[n]_q \varphi_n ([n]_q^{\beta-1} x)} x = \frac{1}{\varphi_n ([n]_q^{\beta-1} x)} \sum_{k=1}^{\infty} \frac{[k]_q}{[n]_q^k q^{k-1} / [k]_q !} D_q^{k} \varphi_n(0)$
which gives the (7). We use a similar technique to get (8). Again differentiating (9) with respect to \( x \) we have

\[
(n_q^\beta - 1)^2 D_q^2 \varphi_n (n_q^\beta - 1 x)
\]

\[
= \sum_{k=2}^{\infty} \frac{[k]_q [k-1]_q}{[k]_q !} ([n_q^\beta - 1 x]^{2} q^{(k-2)(k-3)/2} ([n_q^\beta - 1 x])^{k-2} D_q^k \varphi_n(0).
\]

Using the equality \([k-1]_q = ([k]_q - q^{k-1})\) and multiplying both sides by \( x^2 \) we have

\[
(n_q^\beta - 1 x)^2 D_q^2 \varphi_n (n_q^\beta - 1 x)
\]

\[
= \sum_{k=1}^{\infty} \frac{[k]_q^2}{[k]_q !} q^{(k-2)(k-3)/2} ([n_q^\beta - 1 x])^k D_q^k \varphi_n(0)
\]

\[
- \sum_{k=1}^{\infty} \frac{[k]_q q^{k-1}}{[k]_q !} q^{(k-1)(k-2)/2} ([n_q^\beta - 1 x])^k D_q^k \varphi_n(0)
\]

\[
= q \sum_{k=1}^{\infty} \left( \frac{[k]_q}{q^{k-1}} \right)^2 q^{(k-1)/2} ([n_q^\beta - 1 x])^k \frac{D_q^k \varphi_n(0)}{[k]_q !}
\]

\[
- q \sum_{k=1}^{\infty} \frac{[k]_q q^{k-1}}{[k]_q !} q^{(k-1)/2} ([n_q^\beta - 1 x])^k \frac{D_q^k \varphi_n(0)}{[k]_q !}
\]

Dividing by \([n_q^{2\beta}] \varphi_n (n_q^{\beta - 1} x)\) we write

\[
\frac{(n_q^\beta - 1 x)^2 D_q^2 \varphi_n (n_q^\beta - 1 x)}{[n_q^{2\beta}] \varphi_n (n_q^{\beta - 1} x)}
\]

\[
= \frac{q}{\varphi_n (n_q^{\beta - 1} x)} \sum_{k=1}^{\infty} \left( \frac{[k]_q}{[n_q^\beta - 1 x]^{k-1}} \right)^2 q^{(k-1)/2} ([n_q^\beta - 1 x])^k \frac{D_q^k \varphi_n(0)}{[k]_q !}
\]

\[
- \frac{q}{[n_q^\beta \varphi_n (n_q^{\beta - 1} x)]} \sum_{k=1}^{\infty} \frac{[k]_q q^{k-1}}{[n_q^\beta - 1 x]^{k-1}} q^{(k-1)/2} ([n_q^\beta - 1 x])^k \frac{D_q^k \varphi_n(0)}{[k]_q !}
\]

which gives the (8) by using formulas (5) and (10).

\[\square\]

3. **A-Statistical convergence in weighted spaces**

Let \( \rho \) denotes a continuous weight function with \( \rho(x) \geq 1, x \in [0, \infty) \) and \( \rho(x) \to \infty \) as \( x \to \infty \). Let \( B_\rho \) be the weighted space of all functions \( f \) defined on the \( \mathbb{R}_+ \) satisfying the condition \( |f(x)| \leq M_f \rho(x) \) with some constant \( M_f \), depending only on \( f \). By \( C_\rho \), let us denote the subspace of
all continuous functions belong to $B_\rho$. Also, let $C_\rho^0$ be the subspace of all functions $f \in C_\rho$ for which $\lim_{|x| \to \infty} f(x)/\rho(x) = 0$. Endowed with the norm $\|f\|_\rho = \sup_{x \geq 0} (|f(x)|/\rho(x))$ these spaces are Banach spaces. Note that the weighted Korovkin type theorem were proved by A.D. Gadjiev [10, 11]. Using $A$-statistical convergence, the weighted Korovkin type theorem was given in [6].

Let $\{L_{n,q}\}$ be the sequence of linear positive operators defined by (5). Then it is easily seen that $L_{n,q} : C_\rho \to B_\rho$.

Let $q = \{q_n\}$ be a sequence satisfying the following conditions

\begin{equation}
\tag{12} \text{st}_A - \lim_{n} q_n = 1 \quad \text{and} \quad \text{st}_A - \lim_{n} q_n^2 = a, \quad (0 \leq a < 1).
\end{equation}

The condition (12) guaranties that $\text{st}_A - \lim_{n} \left( |n|^{-1}_q \right) = 0$.

Now we are ready to prove our first result which is related to the $A$-statistical convergence the sequence of $\{L_{n,q}(f)\}$ to $f$.

**Theorem 3.1.** Let $A = (a_{jn})$ be non-negative regular summability matrix, the sequence $q = \{q_n\}$ satisfies (12) with $q_n \in (0, 1]$ for all $n \in \mathbb{N}$. Then for every $f \in C_\rho^0 [0, \infty)$, $\text{st}_A - \lim_{n} \|L_{n,q}(f) - f\|_\rho = 0$ where $\rho(x) = 1 + x^2$.

**Proof.** From Lemma 1, it is obvious that $\text{st}_A - \lim_{n} \|L_{n,q}(e_0) - e_0\|_\rho = 0$. Using the (6), we get

\[
\frac{|L_{n,q}(e_1)(x) - e_1(x)|}{1 + x^2} = \frac{x}{1 + x^2} \frac{|D_{q_n} \varphi_n \left( \left[ n \right]^{-1}_q \beta^{-1} x \right) - 1|}{\left| \frac{D_{q_n} \varphi_n \left( \left[ n \right]^{-1}_q \beta^{-1} x \right) - 1}{\varphi_n \left( \left[ n \right]^{-1}_q \beta^{-1} x \right)} \right|} \leq \|e_1\|_\rho \frac{\left| D_{q_n} \varphi_n \left( \left[ n \right]^{-1}_q \beta^{-1} x \right) - 1 \right|}{\left| \frac{\varphi_n \left( \left[ n \right]^{-1}_q \beta^{-1} x \right) - 1}{\varphi_n \left( \left[ n \right]^{-1}_q \beta^{-1} x \right)} \right|} = B_{n,q}(\varphi, x).
\]

Now, for a given $\varepsilon > 0$, we define the sets $U = \left\{ n : \|L_{n,q}(e_1) - e_1\|_\rho \geq \varepsilon \right\}$ and $U_1 = \left\{ n : B_{n,q}(\varphi_n, x) \geq \varepsilon \right\}$. It is clear that $U \subset U_1$ and hence

\[
\delta_A \left\{ n \in \mathbb{N} : \|L_{n,q}(e_1) - e_1\|_\rho \geq \varepsilon \right\} \leq \delta_A \left\{ n \in \mathbb{N} : B_{n,q}(\varphi_n, x) \geq \varepsilon \right\}.
\]

From the condition (3) we get $\text{st}_A - \lim B_{n,q}(\varphi_n, x) = 0$. Therefore, it is clear that

\[
\delta_A \left\{ n \in \mathbb{N} : B_{n,q}(\varphi_n, x) \geq \varepsilon \right\} = 0
\]

and hence we have

\[
\delta_A \left\{ n \in \mathbb{N} : \|L_{n,q}(e_1) - e_1\|_\rho \geq \varepsilon \right\} = 0,
\]
which implies
\[ st_A - \lim \|L_{n,q_n} (e_1) - e_1\|_\rho = 0. \]

Similarly from (8) we can write
\[ |L_{n,q_n} (e_2) - e_2| = \left( \frac{D_{q_n}^2 \varphi_n \left( \lfloor n \beta - 1 \rfloor x \right)}{q_n \lfloor n \beta - 1 \rfloor \varphi_n \left( \lfloor n \beta - 1 \rfloor x \right)} - 1 \right) x^2 \]
\[ + \frac{1}{\lfloor n \beta \rfloor} \left( \frac{D_{q_n} \varphi_n \left( \lfloor n \beta - 1 \rfloor x \right)}{\lfloor n \beta \rfloor \varphi_n \left( \lfloor n \beta - 1 \rfloor x \right)} - 1 \right) x + \frac{1}{\lfloor n \beta \rfloor} x \]

and hence we get
\[ \frac{|L_{n,q_n} (e_2) - e_2|}{1 + x^2} \leq \|e_2\|_\rho \left| \frac{D_{q_n}^2 \varphi_n \left( \lfloor n \beta - 1 \rfloor x \right)}{q_n \lfloor n \beta - 1 \rfloor \varphi_n \left( \lfloor n \beta - 1 \rfloor x \right)} - 1 \right| \]
\[ + \frac{1}{\lfloor n \beta \rfloor} \|e_1\|_\rho \left| \frac{D_{q_n} \varphi_n \left( \lfloor n \beta - 1 \rfloor x \right)}{\lfloor n \beta \rfloor \varphi_n \left( \lfloor n \beta - 1 \rfloor x \right)} - 1 \right| + \frac{1}{\lfloor n \beta \rfloor} \|e_1\|_\rho \]
\[ \leq B_{n,q_n} (\varphi_n, x) + \frac{1}{\lfloor n \beta \rfloor} C_{n,q} (\varphi_n, x) + \frac{1}{\lfloor n \beta \rfloor}, \]

Now for given \( \varepsilon > 0 \), let us define the following sets
\[ V = \{ n : \|L_{n,q_n} (e_2) - e_2\|_\rho \geq \varepsilon \}, \]
\[ V_1 = \{ n : B_{n,q_n} (\varphi_n, x) \geq \varepsilon / 3 \}, \]
\[ V_2 = \{ n : C_{n,q_n} (\varphi_n, x) \geq \varepsilon / 3 \}, \]
\[ V_3 = \{ n : \lfloor n \beta \rfloor \geq \varepsilon / 3 \}. \]

It is obviously that \( V \subset V_1 \cup V_2 \cup V_3 \). From the condition (12) we get
\[ st_A - \lim_{n \to \infty} \frac{1}{\lfloor n \beta \rfloor} = st_A - \lim_{n \to \infty} ((1 - q_n) / (1 - q_n^n))^\beta = 0 \]

with \( 0 < \beta \leq 2/3 \). Hence by the condition (3) we have \( st_A - \lim B_{n,q_n} (\varphi_n, x) = 0 \) and \( st_A - \lim C_{n,q_n} (\varphi_n, x) = 0 \). Then we obtain \( \delta_A (V_k) = 0, \ k = 1, 2, 3 \). Since \( \delta_A (V) \leq \delta_A (V_1) + \delta_A (V_2) + \delta_A (V_3) \) we find that \( st_A - \lim \|L_{n,q_n} (e_2) - e_2\|_\rho = 0 \).

Consequently we obtain that \( st_A - \lim \|L_{n,q_n} (e_i) - e_i\|_\rho = 0, i = 0, 1, 2 \) which completes the proof of the Theorem according to the weighted Ko-rovkin type Theorem [12, 6, 10]. \( \square \)

As a consequence, for all \( n \in \mathbb{N}, x \geq 0 \) and \( 0 < q_n < 1 \), we have
\[ st_A - \lim_n \|L_{n,q_n} ((e_1 - e_0 x)^\nu)\|_\rho = 0, \nu = 1, 2 \]
where

\begin{equation}
L_{n,q_n} \left( (e_1 - e_0 x) \right) (x) = \left( D_{q_n} \varphi_n \left( \frac{[n]_{q_n}^{\beta-1} x}{[n]_{q_n} \varphi_n([n]_{q_n}^{\beta-1} x)} - 1 \right) \right) x,
\end{equation}

\begin{equation}
L_{n,q_n} \left( (e_1 - e_0 x)^2 \right) (x) = \left( \frac{D^2_{q_n} \varphi_n \left( \frac{[n]_{q_n}^{\beta-1} x}{[n]_{q_n}^{2} \varphi_n([n]_{q_n}^{\beta-1} x)} \right) - 2 \frac{D_{q_n} \varphi_n \left( \frac{[n]_{q_n}^{\beta-1} x}{[n]_{q_n} \varphi_n([n]_{q_n}^{\beta-1} x)} + 1 \right) x}{[n]_{q_n}^{\beta}} \right)
\end{equation}

\begin{equation}
+ \frac{1}{[n]_{q_n}^{\beta}} \frac{D_{q_n} \varphi_n \left( \frac{[n]_{q_n}^{\beta-1} x}{[n]_{q_n} \varphi_n([n]_{q_n}^{\beta-1} x)} - 1 \right) x + \frac{1}{[n]_{q_n}^{\beta}} x. \nonumber
\end{equation}

**Theorem 3.2.** Let $A = (a_{j,n})$ be non-negative regular summability matrix, the sequence $q = \{q_n\}$ satisfies (12) with $q_n \in (0, 1)$ for all $n \in \mathbb{N}$. If any function $f \in C_p$, satisfies the Lipschitz condition that is

$$
|f(t) - f(x)| \leq M |t - x|^\alpha, \quad 0 \leq \alpha < 1, \quad x, t \geq 0
$$

then

$$
st_A - \limsup_{n} \frac{|L_{n,q_n} (f) (x) - f(x)|}{1 + x^\alpha} = 0
$$

where $M$ is a constant.

**Proof.** Since $L_{n,q_n}$ is a linear positive operator and $f$ satisfies the Lipschitz condition we can write,

$$
|L_{n,q_n} (f) (x) - f(x)| \leq L_{n,q_n} (|f(t) - f(x)|) (x)
$$

$$
\leq \frac{M}{\varphi_n([n]_{q_n}^{\beta-1} x)} \sum_{k=0}^{\infty} \left| \frac{[k]_{q_n}}{[n]_{q_n}^{\beta} q_n^{k-1} - x} \right|^\alpha \left( \frac{[n]_{q_n}^{\beta-1} x}{k_{q_n}!} \right)^k D_{q_n} \varphi_n(0).
$$

Applying the Holder inequality with $p = 2/\alpha, s = 2/(2 - \alpha)$ and saying $B_{n,q_n,k} (\varphi_n; x) := \varphi_n^{-1} \left( \frac{[n]_{q_n}^{\beta-1} x}{[n]_{q_n}!} \right)^k \frac{D_{q_n} \varphi_n(0)}{k_{q_n}!}$, from Lemma 1 we get

$$
|L_{n,q_n} (f) (x) - f(x)| \leq M \left( \sum_{k=0}^{\infty} \left( \frac{[k]_{q_n}}{[n]_{q_n}^{\beta} q_n^{k-1} - x} \right)^2 B_{n,q_n,k} (\varphi_n; x) \right)^{\alpha/2}
$$

$$
\times \left( \sum_{k=0}^{\infty} B_{n,q_n,k} (\varphi_n; x) \right)^{(2-\alpha)/2}
$$

$$
= M \left( L_{n,q_n} \left( (e_1 - e_0 x)^2 \right) (x) \right)^{\alpha/2}.
$$
Taking account to (16) and using the conditions (3) and (12), similarly with the proof of the theorem 1, we obtain the desired result. □

Now, we concern with the order of approximation of a function \( f \in C^0_\rho \) by the linear positive operator \( L_{n,q} \). We will use the weighted modulus of continuity defined by

\[
\Omega_m(f; \delta) = \sup_{x \in [0, \infty), |h| \leq \delta} \frac{|f(x + h) - f(x)|}{1 + (x + h)^m},
\]

for each \( f \in C^0_\rho \), \( \rho(x) = 1 + x^m \), \( x \in [0, \infty), m \in \mathbb{N} \).

The weighted modulus of continuity has the following properties (see [14]):

(i) \( \lim_{\delta \to 0} \Omega_m(f; \delta) = 0 \) for each \( f \in C^0_\rho \)

(ii) \( \Omega_m(f; \lambda \delta) \leq (\lambda + 1) \Omega_m(f; \delta) \) for each positive real number \( \lambda, m \in \mathbb{N} \)

(iii) \(|f(t) - f(x)| \leq (1 + (x + |t - x|)^m) \left( \frac{|t - x|}{\delta} + 1 \right) \Omega_m(f; \delta) \) for every \( x, t \in [0, \infty), m \in \mathbb{N} \).

Notice that, if \( f \) is not uniformly continuous on the interval \([0, \infty)\); then the usual first modulus of continuity \( \omega(f; \delta) \) does not tend to zero, as \( \delta \to 0 \). It is seen that \( \Omega_m(f; \delta) \to 0 \) as \( \delta \to 0 \) for all \( f \in C^0_\rho \) due to the property (i).

We now give second our main result. The following theorem is given an estimate for the approximation error with the operators \( L_{n,q_n}(f) \), by means of \( \Omega_1(f; \delta) \) with \( \rho(x) = 1 + x \).

**Theorem 3.3.** Let \( \{q_n\} \) be a sequence satisfying the condition (12) with \( q_n \in (0, 1) \) for all \( n \in \mathbb{N} \). Suppose that the condition

\[
\left( \frac{D^\nu_{q_n} \varphi_n([n]^{\beta-1}_{q_n} x)}{[q_n]^{\nu-1} \varphi_{n_q}([n]^{\beta-1}_{q_n} x)} - 1 \right) = O \left( 1/ [n]^{\beta}_{q_n} \right)
\]

holds instead of (3). If \( f \in C^0_\rho \) with \( \rho(x) = 1 + x \) then the inequality

\[
||L_{n,q_n}(f)(x) - f(x)||_{\rho_2} \leq C \Omega_1 \left( f; 1/ \sqrt{[n]^{\beta}_{q_n}} \right) \left( 1 + 1/ [n]^{\beta}_{q_n} \right)
\]

holds where \( \rho_2(x) = 1 + x^2 \) and \( C \) is a constant independent of \( f \) and \( n \).

**Proof.** Considering the definition of \( \Omega_1(f; \delta) \) and by using the property (iii) of \( \Omega_1(f; \delta) \) we can write

\[
|f(t) - f(x)| \leq (1 + x + |t - x|) \left( \frac{|t - x|}{\delta} + 1 \right) \Omega_1(f; \delta)
\]

\[
\leq (1 + 2x + t) \left( \frac{|t - x|}{\delta} + 1 \right) \Omega_1(f; \delta).
\]
Since $L_{n,q_n}$ is a linear and positive operator we get
\[
|L_{n,q_n} (f) (x) - f (x)| \leq L_{n,q_n} (|f(t) - f(x)|) (x)
\]
(18)
\[
\leq \Omega_1(f; \delta) \left[ L_{n,q_n} (1 + 2x + t) (x)
+ L_{n,q_n} \left( (1 + 2x + t) \left| \frac{t-x}{\delta} \right| \right) (x) \right].
\]

To estimate the first term, considering (6) and (7), we can write
\[
(L_{n,q_n} ((1 + 2x + t)) (x))
= (1 + 2x) L_{n,q_n} (e_0) (x) + L_{n,q_n} (e_1) (x)
\]
(19)
\[
= (1 + 3x) + \left( \frac{1}{[n]_{q_n}^\beta} D_{q_n} \varphi_n \left( \frac{[n]_{q_n}^{\beta-1} x}{[n]_{q_n}^\beta} \right) - 1 \right) x
\]
\[
\leq 3 (1 + x) \left[ 1 + O \left( \frac{1}{[n]_{q_n}^\beta} \right) \right].
\]

Applying the Cauchy-Schwarz inequality to the second term in (18), since $L_{n,q_n}$ is a linear and positive, we get
\[
L_{n,q_n} \left( (1 + 2x + t) \left| \frac{t-x}{\delta} \right| \right) (x)
\]
(20)
\[
\leq \left( L_{n,q_n} \left( (1 + 2x + t)^2 \right) (x) \right)^{1/2} \times \left( L_{n,q_n} \left( \frac{(t-x)^2}{\delta^2} \right) (x) \right)^{1/2}.
\]

We now estimate the first term. By using (6),(7) and (8) and by simple calculations ,we get
\[
(L_{n,q_n} \left( (1 + 2x + t)^2 \right) (x) \right)^{1/2} \leq 4 (1 + x) \left[ 1 + O \left( \frac{1}{[n]_{q_n}^\beta} \right) \right]^{1/2}.
\]
(21)

Taking into account (16), if we estimate the second term then we get
\[
(L_{n,q_n} \left( \frac{|t-x|^2}{\delta^2} \right) (x) \right)^{1/2} = \frac{1}{\delta} \left( L_{n,q_n} \left( (e_1 - e_0x)^2 \right) (x) \right)^{1/2}
\]
(22)
\[
= \frac{1}{\delta} \left( O \left( \frac{1}{[n]_{q_n}^\beta} \right) (x^2 + x) \right)^{1/2}
\]
\[
\leq \frac{1}{\delta} \left( O \left( \frac{1}{[n]_{q_n}^\beta} \right) (1 + x)^2 \right)^{1/2}
\]
\[
\leq \frac{1}{\delta} (1 + x) \sqrt{C_1} \frac{1}{[n]_{q_n}^\beta}.
\]
with $C_1$ is a constant independent of $n$. Combining (19), (20), (21) and (22) with (18) we have
\[
|L_{n,q_n}(f)(x) - f(x)| \leq \Omega_1(f;\delta) \times C_2 (1 + x)^2 \left[ 1 + O \left( \frac{1}{[n]_{q_n}^{\beta}} \right) \right] \alpha \sqrt{\frac{1}{[n]_{q_n}^{\beta}}},
\]
where $C_2 = \max\{3,4C_1\}$. Taking $\delta := \delta_n = \left( \frac{1}{[n]_{q_n}^{\beta}} \right)^{1/2}$ we obtain
\[
|L_{n,q_n}(f)(x) - f(x)| \leq 2C_2\Omega_1(f;\delta_n) (1 + x^2) \left[ C_3 + O \left( \frac{1}{[n]_{q_n}^{\beta}} \right) \right] \delta
\]
with $C_3 = \sup_{x \geq 0} \frac{(1+x)^2}{1+x^2}, C = \max\{2C_2, C_3, C_3 C_1\}$ which gives that the (17).

We notice that, from (13), it is clear that $\text{st}_A - \lim_n \delta_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore $\text{st}_A - \lim_n \Omega_1(f;\delta_n) \rightarrow 0$ due to the property (i) of $\Omega_1(f;\delta)$. Consequently, order of $A$-statistical convergence of the sequence of $\{L_{n,q_n}(f)\}$ to $f$ is $\left( \frac{1}{[n]_{q_n}^{\beta}} \right)^{1/2}$ in the $\rho_2$-norm. $\square$

4. LOCAL APPROXIMATION

**Theorem 4.1.** Let $\{q_n\}$ be a sequence satisfying the condition (12) with $q_n \in (0,1]$ for all $n \in \mathbb{N}$. We have

1) For any $f \in C_\rho$ we have
\[
|L_{n,q_n}(f)(x) - f(x)| \leq 2\omega(f;\sqrt{\delta_n x})
\]
where $\omega(f,\delta)$ is the usual first modulus of continuity of $f$ and
\[
\delta_{n,x} = L_{n,q_n} \left( (e_1 - e_0 x)^2 \right)(x)
\]
and $\text{st}_A - \lim_n \delta_{n,x} = 0$, for all fixed $x \in [0,\infty)$.

2) If $f \in C_\rho$ satisfies the Lipschitz condition then $|L_{n,q_n}(f)(x) - f(x)| \leq M\delta_{n,x}^{\alpha/2}$, $0 \leq \alpha < 1$.

**Proof.** 1) Using the linearity and positivity of the operator $L_{n,q_n}$ and the known properties of $\omega(f,\delta)$ and applying Cauchy -Schwarz inequality we obtain
\[
|L_{n,q_n}(f)(x) - f(x)| \leq L_{n,q_n}(|f(t) - f(x)|)(x)
\]
\[
\leq \omega(f;\delta) \left[ L_{n,q_n}(e_0)(x) + \frac{1}{\delta} \left( L_{n,q_n}(e_1 - e_0 x)^2(x) \right)^{1/2} \right].
\]

By choosing $\delta = \sqrt{\delta_{n,x}}$ as in (23), we reach the desired result. Notice that, taking into account (14), we get $\text{st}_A - \lim_n \delta_{n,x} = 0$ for all fixed $x$. 

Hence we have \( st_A - \lim_n \omega(f; \delta_{n,x}) = 0 \). This gives the pointwise rate of \( A \)-statistical convergence of the operator \( L_{n,q_n}(f) \) to the function \( f \).

2) Considering the proof of Theorem 2 and formula (23) we obtain that

\[
|L_{n,q_n}(f)(x) - f(x)| \leq M \left( L_{n,q_n}\left((e_1 - e_0x)^2\right)(x)\right)^{\alpha/2} = M\delta_{n,x}^{\alpha/2}.
\]

\[\square\]

Now we give the rate of \( A \)-statistical convergence for the operators \( L_{n,q_n}(f) \) by using the Peetre’s \( K \)-functional in the space \( C_B^2[0, \infty) \).

Let \( C_B[0, \infty) \) be the space of all real valued uniformly continuous and bounded functions \( f \) on the interval \([0, \infty)\) with the norm

\[
\|f\|_{C_B} = \sup_{x \in [0, \infty)} |f(x)|.
\]

The Peetre’s \( K \)-functional of function \( f \in C_B[0, \infty) \) is defined by

\[
K(f; \delta) = \inf_{g \in C_B^2} \left\{ \|f - g\|_{C_B} + \delta \|g\|_{C_B^2} \right\}
\]

where \( \delta > 0 \) and \( C_B^2[0, \infty) = \{f \in C_B : f', f'' \in C_B[0, \infty)\} \) endowed with the norm

\[
\|f\|_{C_B^2} = \|f\|_{C_B} + \left\| f' \right\|_{C_B} + \left\| f'' \right\|_{C_B}.
\]

**Theorem 4.2.** For each \( f \in C_B^2[0, \infty) \) we have

\[
st_A - \lim_n \|L_{n,q_n}f - f\|_{C_B} = 0.
\]

**Proof.** Applying the Taylor expansion to the function \( f \in C_B^2[0, \infty) \), we can write

\[
L_{n,q_n}(f)(x) - f(x) = f''(x)L_{n,q_n}\left((e_1 - e_0x)\right)(x) + \frac{1}{2} f''(\zeta)L_{n,q_n}\left((e_1 - e_0x)^2\right)(x), \quad \zeta \in (t, x),
\]

where \( L_{n,q_n}\left((e_1 - e_0x)\right)(x), L_{n,q_n}\left((e_1 - e_0x)^2\right)(x) \) are given by (15) and (16) respectively.

Hence

\[
\|L_{n,q_n}(f) - f\|_{C_B} \leq \left\| f' \right\|_{C_B} \|L_{n,q_n}\left((e_1 - e_0x)\right)\|_{C[0, A]} + \left\| f'' \right\|_{C_B} \|L_{n,q_n}\left((e_1 - e_0x)^2\right)\|_{C[0, A]}.
\]

(25)

Now for given \( \varepsilon > 0 \), let us define \( U = \left\{ n \in \mathbb{N} : \|L_{n,q_n}(f) - f\|_{C[0, \alpha]} \geq \varepsilon \right\} \),

\[
U_1 = \left\{ n \in \mathbb{N} : \left\| f' \right\|_{C_B} \|L_{n,q_n}\left((e_1 - e_0x)\right)\|_{C[0, \alpha]} \geq \varepsilon/2 \right\},
\]
\[ U_2 = \left\{ n \in \mathbb{N} : \| f'' \|_{C_B} \left\| L_{n,q_n} \left( (e_1 - e_0 x)^2 \right) \right\|_{C[0,\alpha]} \geq \varepsilon / 2 \right\}. \]

It is obvious that \( U \subset U_1 \cup U_2 \) and hence \( \delta_A U \leq \delta_A U_1 + \delta_A U_2 \). By using (14) we get \( \lim_{n} \| L_{n,q_n} \left( (e_1 - e_0 x)^\nu \right) \|_{C[0,\alpha]} = 0, \nu = 1, 2 \) with \([0, \alpha] \subset [0, \infty)\) and \( \| \cdot \|_{C[0,\alpha]} \) is maximum norm. Therefore we obtain \( \delta_A U_1 = 0, \delta_A U_2 = 0 \) so \( \delta_A U = 0 \) which completes the proof. \( \square \)

**Theorem 4.3.** For each \( f \in C_B [0, \infty) \)
\[
\| L_{n,q_n} (f) - f \|_{C_B} \leq K (f; \delta_{n,x})
\]
where \( \{ K (f; \delta_{n,x}) \} \) is the sequence of Peetre’s \( K \)-functional and
\[
\delta_{n,x} = \| L_{n,q_n} (e_1 - e_0 x) \|_{C[0,\alpha]} + \left\| L_n (e_1 - e_0 x)^2 \right\|_{C[0,\alpha]}
\]
and \( \lim_{n} \delta_{n,x} = 0 \) for each fixed \( x \in [0, \infty) \).

**Proof.** For each \( g \in C^2_B [0, \infty) \), by using (24) and (25), we get
\[
\| L_{n,q_n} g - g \|_{C^2_B} \leq \left( \| L_{n,q_n} (e_1 - e_0 x) \|_{C[0,\alpha]} \right) \| g \|_{C^2_B}
\]
\[
+ \left\| L_{n,q_n} (e_1 - e_0 x)^2 \right\|_{C[0,\alpha]} \| g \|_{C^2_B}
= \delta_{n,x} \| g \|_{C^2_B} \text{ say.}
\]

For each \( f \in C_B [0, \infty) \) and \( g \in C^2_B [0, \infty) \), we obtain
\[
\| L_{n,q_n} f - f \|_{C^2_B} \leq \| L_{n,q_n} f - L_{n,q_n} g \|_{C_B} + \| L_{n,q_n} g - g \|_{C^2_B} + \| g - f \|_{C_B}
\]
\[
\leq 2 \| g - f \|_{C_B} + \| L_{n,q_n} g - g \|_{C^2_B}
\]
\[
\leq 2 \| g - f \|_{C_B} + \delta_{n,x} \| g \|_{C^2_B}
\]
\[
\leq 2 \left( \| g - f \|_{C_B} + \delta_{n,x} \| g \|_{C^2_B} \right).
\]

Taking the infimum on the right hand side over all \( g \in C^2_B [0, \infty) \) we get
\[
\| L_{n,q_n} f - f \|_{C^2_B} \leq K (f; \delta_{n,x}).
\]

By (14), we get \( \lim_{n} \delta_{n,x} = 0 \) so \( \lim_{n} K (f; \delta_{n,x}) = 0 \). Therefore we obtain the rate of \( A \)-statistical convergence of the sequence of the operators \( L_{n,q_n} (f) \) to \( f \) in the space \( C_B [0, \infty) \). \( \square \)

5. **Concluding Remarks**

Some particular cases of the operators \( L_{n,q} \) are defined as follows:
If we take $\phi_n(x) = (1 + x)^n$ then we obtain $q$-Balazs-Szabados operators which are studied by O. Dogru [5]. In [5], the function $f$ has been taken as $f\left(\left\lfloor k \right\rfloor_q / \left\lfloor n \right\rfloor_q^\beta \right)$ instead of $f\left(\left\lfloor k \right\rfloor_q / q^{k-1} \left\lfloor n \right\rfloor_q^\beta \right)$ which is a natural generalization of $q$-Balazs-Szabados operators.

b) Taking into account the $q$-analogues of the exponential function given by $E_q\left(\left\lfloor n \right\rfloor_q x \right) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \left(\left\lfloor n \right\rfloor_q^\beta x \right)^k$ and $e_q\left(\left\lfloor n \right\rfloor_q x \right) = \sum_{k=0}^{\infty} \frac{\left(\left\lfloor n \right\rfloor_q x \right)^k}{\left[ k \right]_q!}$ choosing $\phi_n(x) = E_q\left(-\left\lfloor n \right\rfloor_q x \right)$ or $\phi_n(x) = e_q\left(-\left\lfloor n \right\rfloor_q x \right)$, with $\beta = 1$, we obtain $q$-Szasz-Mirakjan operators studied in different spaces in [1] and [20], respectively.

c) Taking $\phi_n(x) = (1 + q^{n-1}x)_q^{-n}$ we obtain the $q$-analogue of classical Baskakov operators studied in [2].

Consequently the $A$–statistical approximation properties are valid in large spectrum of the operators (5).

If we take $A = I$, identity matrix, we have the ordinary rate of convergence for the operators (5) (see, [14, 16]).

References


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**NURHAYAT İSPIR**

**DEPARTMENT OF MATHEMATICS**
**FACULTY OF SCIENCES**
**GAZI UNIVERSITY**
**06500 ANKARA**
**TURKEY**

_E-mail address: nispir@gazi.edu.tr_