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On a General Class of q-Rational Type Operators

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ABSTRACT. In this study, we define a general class of rational type operators based on q-calculus and investigate the weighted approximation properties of these operators by using A-statistical convergence. We also estimate the rates of A-statistical convergence of these operators by modulus of continuity and Petree's K-functional. The operators to be introduced, include some well known q-operators so our results are true in a large spectrum of these operators.

1. INTRODUCTION

K. Balazs [3] introduced the Bernstein type rational functions and proved the convergence theorems for them. Later, K.Balazs and J.Szabados [4] improved some estimates on the order of approximation for the Bernstein type rational operators.

The generalization of the Bernstein type rational operators are introduced by C. Atakut and N. Ispir [16] as follows

$$L_n(f)(x) = \frac{1}{\psi_n(a_n x)} \sum_{k=0}^{\infty} f\left(\frac{k}{b_n}\right) \frac{\psi_n^{(k)}(0)}{k!} (a_n x)^k, \quad n \in \mathbb{N},\ x \geq 0, \tag{1}$$

where a_n and b_n are suitably chosen real numbers, independent of x. Here $\{\psi_n\}$ is a sequence of functions $\psi_n : \mathbb{C} \to \mathbb{C}$ satisfying the following conditions:

a) ψ_n $(n = 1, 2, \dots)$ is analytic on a domain D containing the disk

$$B = \{z \in \mathbb{C} : |z - b| \leq b\};$$

b) $\psi_n(0) = 1, \quad (n = 1, 2, \dots)$;

c) For any $x \geq 0$, $\psi_n(x) > 0$ and $\psi_n^{(k)}(0) \geq 0$ for any $n = 1, 2, \dots$, $k = 1, 2, \dots$

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d) For every $n = 1, 2, \dots,$ it is

(2) $$\lim_n \left(\frac{\psi_n^{(\nu)}(a_n x)}{n^\nu \psi_n(a_n x)} - 1 \right) = 0; \quad \nu = 1, 2,$$

where $a_n \to 0$, as $n \to \infty$.

In [16] the authors estimated the order of approximation for the operators defined by (1) and proved a Voronovskaja type asymptotic formula and pointwise convergence in simultaneous approximation. In [13, 14, 18] the approximation properties for different variants of the operators (1) were investigated in various function spaces . In [17], the approximation properties of the Kantorovich variant of the operators (1) were given by the aid of A-statistical convergence. Notice that A -statistical convergence is stronger than usual convergence.

It is known that the applications of q-calculus in the area of approximation theory have been an active area of research. In the recent years, the statistical approximation properties of some positive linear operators based on q-integers have been studied intensively by many authors (e.g.[5, 15, 17, 21]).

The aim of this study is to introduce q-type generalization of the operators (1) and investigate the A-statistical approximation properties of the constructed operators in weighted spaces. Using A-statistical convergence, we obtain weighted Korovkin type theorem and weighted order of approximation by the constructed operators based q-calculus. Moreover, we estimate the rate of A-statistical convergence by usual modulus of continuity and by Petree's K-functional in the different normed spaces for q-extension of the operators (1).

Now, let us give a few basic definitions and notations in q-integers shortly. Details on q-calculus can be found in [7]. Throughout the present paper, we consider q as a real number such that $0 < q < 1$, and for each nonnegative integer i, the q-integer $[i]_q$ is defined by $[i]_q = \left(1 - q^i\right) / (1 - q)$, $[0]_q := 0$; and q-factorial $[i]_q!$ is defined by $[i]_q! = [1]_q [2]_q \dots [i]_q$, $[0]! := 1$.

For fixed $0 < q < 1$, the q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ with respect to x is defined by $D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$, $x \neq 0$ and $D_q f(0) = \lim_{x \to 0} D_q f(x)$. The chain rule for ordinary derivatives is similar for q-derivative.

At this point, we recall the q-Taylor theorem in the following.

Theorem A([7], p. 103.) If a function $f(x)$ possess convergence series expansion then

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)_q^n}{[n]_q!} D_q^n [f(a)]$$

where $(x-a)_q^n = \prod_{s=0}^{n-1} (x - q^s a) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)} x^{n-k} (-a)^k$.

Now let us recall some concepts of the A-statistical convergence. Suppose that A is non-negative summability matrix and let K be subset of $\mathbb{N}$

the set of natural numbers. The $A-$density of K is defined by $\delta_A(K) := \lim_j \frac{1}{n}\sum_{n=1}^{\infty} a_{jn}\chi_K(n)$ provided limit exists, where χ_K characteristic function of K. A sequence $x = (x_n)$ is called $A-$statistically convergent to L if for every $\varepsilon > 0$ $\lim_j \sum_{n:\,|x_n-L|\geq\varepsilon} a_{jn} = 0$ or equivalently for every $\varepsilon > 0$, $\delta_A\{k \in \mathbb{N}\colon |x_k - L| \geq \varepsilon\} = 0$. In this case we write $st_A - \lim x = L$ [8, 9].

The case in which $A = C_1$, the Cesáro matrix of order one, A -statistical convergence reduces to the statistical convergence [9, 19]. Also if $A = I$, the identity matrix, then it reduces to the ordinary convergence. We note that, if $A = (a_{jn})$ is a non-negative regular matrix such that $\lim_j \max_n \{a_{jn}\} = 0$, then A-statistical convergence is stronger than convergence [19]. It should be noted that the concept of A -statistical convergence may also be given in normed spaces: Assume $(X, \|.\|)$ is a normed space and $u = (u_k)$ is a X-valued sequence. Then (u_k) is said to be A-statistically convergent to $u_0 \in X$ if, for every $\varepsilon > 0$, $\delta_A\{k \in \mathbb{N} : \|u_k - u_0\| \geq \varepsilon\} = 0$ [19].

2. Construction of operators and auxiliary results

Now we would like to introduce q-generalization of the operators (1).

Let (φ_n) be a sequence of real functions on $\mathbb{R}_+$ which are continuously infinitely $q-$differentiable on $\mathbb{R}_+$ satisfying the following conditions

1. $\varphi_n(0) = 1$, for each $n \in \mathbb{N}$
2. $D_q^k\ \varphi_n(0) \geq 0$, for every $n, k \in \mathbb{N}$, $x \geq 0$
3. For every $n \in \mathbb{N}$, with $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$, $q \in (0, 1]$, $0 < \beta \leq \frac{2}{3}$

$$st_A - \lim_n \left(\frac{D_q^{\nu}\varphi_n([n]_q^{\beta-1}\, x)}{q^{\nu-1}\,[n]_q^{\nu}\,\varphi_n([n]_q^{\beta-1}\, x)} - 1 \right) = 0\,; \quad \nu = 1, 2. \tag{3}$$

For fixed $x \in \mathbb{R}_+$, taking account to Theorem A we get

$$\varphi_n\left([n]_q^{\beta-1}\, x\right) = \sum_{k=0}^{\infty} \frac{\left([n]_q^{\beta-1}\, x\right)_q^k}{[k]_q!} D_q^k\ \varphi_n(0)\,. \tag{4}$$

Using the formula $(a+b)_q^n = \prod_{s=0}^{n-1}(a + q^s b) = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k a^{n-k}$ and taking $a = 0, b = [n]_q^{\beta-1}\, x$ we write $\left([n]_q^{\beta-1}\, x\right)_q^k = q^{k(k-1)/2}\left([n]_q^{\beta-1}\, x\right)^k$. Choosing $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$, $0 < \beta \leq \frac{2}{3}, n \in \mathbb{N}$, we introduce q-generalization of the operator (1) as follows

(5)

$$L_{n,q}(f)(x) = \frac{1}{\varphi_n([n]_q^{\beta-1}\, x)} \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n]_q^{\beta}\, q^{k-1}}\right) q^{k(k-1)/2} \frac{([n]_q^{\beta-1}\, x)^k}{[k]_q!} D_q^k \varphi_n(0),$$

for each $n \in \mathbb{N}$, $x \geq 0$.
Note that:

- It is easily verified that the operators $L_{n,q}$ are linear positive operators.
- The order of convergence is the best possible estimate for $\beta \in (0, 2/3]$ (see [4]).
- The present condition (3) is weaker than the present one given by (2) for $q = 1$. Indeed, we can construct a sequence such that it is statistically convergent to 1 but not convergent in the ordinary sense. A well known example is defined as; $\alpha_n = \sqrt{n}$ if $n = m^2$ $(m \in \mathbf{N})$, and $\alpha_n = 1$ otherwise. Same result also works to $A-$statistical convergence.

Lemma 2.1. *For all $n \in \mathbb{N}$, $x \geq 0$ and $0 < q < 1$ we get*

$$L_{n,q}(e_0)(x) = 1, \tag{6}$$

$$L_{n,q}(e_1)(x) = \frac{D_q\varphi_n\left([n]_q^{\beta-1}x\right)}{[n]_q\ \varphi_n([n]_q^{\beta-1}x)}x, \tag{7}$$

$$L_{n,q}(e_2)(x) = \frac{D_q^2\varphi_n\left([n]_q^{\beta-1}x\right)}{q\ [n]_q^2\ \varphi_n([n]_q^{\beta-1}x)}x^2 + \frac{1}{[n]_q^\beta}\frac{D_q\varphi_n\left([n]_q^{\beta-1}x\right)}{[n]_q\ \varphi_n([n]_q^{\beta-1}x)}x \tag{8}$$

where $(e_i)(x) = x^i$, $i = 0, 1, 2$.

Proof. From (4) and definition of $L_{n,q}(f)$ it is clear that $L_{n,q}(e_0)(x) = 1$. Considering (4), we can write the q-derivative of φ_n with respect to x as

$$[n]_q^{\beta-1} D_q\varphi_n\left([n]_q^{\beta-1}x\right) = \sum_{k=1}^{\infty}\frac{[k]_q}{[k]_q!}[n]_q^{\beta-1}\left([n]_q^{\beta-1}x\right)_q^{k-1}D_q^k\varphi_n(0) \tag{9}$$

$$= \sum_{k=1}^{\infty}\frac{[k]_q}{[k]_q!}[n]_q^{\beta-1}q^{(k-1)(k-2)/2}([n]_q^{\beta-1}x)^{k-1}D_q^k\varphi_n(0)$$

where used the equation $(a_nx)_q^k = q^{k(k-1)}(a_nx)^k$. Hence multiplying both sides by x and dividing by $[n]_q^\beta\ \varphi_n\left([n]_q^{\beta-1}x\right)$ we obtain

$$\frac{D_q\varphi_n\left([n]_q^{\beta-1}x\right)}{[n]_q\ \varphi_n\left([n]_q^{\beta-1}x\right)}x = \tag{10}$$

$$= \frac{1}{\varphi_n\left([n]_q^{\beta-1}x\right)}\sum_{k=1}^{\infty}\frac{[k]_q}{[n]_q^\beta q^{k-1}}q^{k(k-1)/2}\frac{([n]_q^{\beta-1}x)^k}{[k]_q!}D_q^k\varphi_n(0)$$

which gives the (7). We use a similar technique to get (8). Again differentiating (9) with respect to x we have

$$(11)\quad \left([n]_q^{\beta-1}\right)^2 D_q^2\varphi_n\left([n]_q^{\beta-1}x\right) = \sum_{k=2}^{\infty}\frac{[k]_q\,[k-1]_q}{[k]_q!}\left([n]_q^{\beta-1}\right)^2 q^{(k-2)(k-3)/2}([n]_q^{\beta-1}x)^{k-2}D_q^k\varphi_n(0).$$

Using the equality $[k-1]_q=\left([k]_q-q^{k-1}\right)$ and multiplying both sides by x^2 we have

$$\begin{aligned}
&\left([n]_q^{\beta-1}x\right)^2 D_q^2\varphi_n\left([n]_q^{\beta-1}x\right)\\
&=\sum_{k=1}^{\infty}\frac{[k]_q^2}{[k]_q!}q^{(k-2)(k-3)/2}([n]_q^{\beta-1}x)^k D_q^k\varphi_n(0)\\
&\quad-\sum_{k=1}^{\infty}\frac{[k]_q}{[k]_q!}q^{k-1}\;q^{(k-1)(k-2)/2}([n]_q^{\beta-1}x)^k D_q^k\varphi_n(0)\\
&=q\sum_{k=1}^{\infty}\left(\frac{[k]_q}{q^{k-1}}\right)^2 q^{k(k-1)/2}([n]_q^{\beta-1}x)^k\frac{D_q^k\varphi_n(0)}{[k]_q!}\\
&\quad-q\sum_{k=1}^{\infty}\frac{[k]_q}{q^{k-1}}q^{k(k-1)/2}([n]_q^{\beta-1}x)^k\frac{D_q^k\varphi_n(0)}{[k]_q!}.
\end{aligned}$$

Dividing by $[n]_q^{2\beta}\;\varphi_n\left([n]_q^{\beta-1}x\right)$ we write

$$\begin{aligned}
&\frac{\left([n]_q^{\beta-1}x\right)^2 D_q^2\varphi_n\left([n]_q^{\beta-1}x\right)}{[n]_q^{2\beta}\;\varphi_n\left([n]_q^{\beta-1}x\right)}\\
&=\frac{q}{\varphi_n\left([n]_q^{\beta-1}x\right)}\sum_{k=1}^{\infty}\left(\frac{[k]_q}{[n]_q^{\beta}\,q^{k-1}}\right)^2 q^{k(k-1)/2}([n]_q^{\beta-1}x)^k\frac{D_q^k\varphi_n(0)}{[k]_q!}\\
&\quad-\frac{q}{[n]_q^{\beta}\;\varphi_n\left([n]_q^{\beta-1}x\right)}\sum_{k=1}^{\infty}\frac{[k]_q}{[n]_q^{\beta}\,q^{k-1}}q^{k(k-1)/2}([n]_q^{\beta-1}x)^k\frac{D_q^k\varphi_n(0)}{[k]_q!}
\end{aligned}$$

which gives the (8) by using formulas (5) and (10). □

3. A-STATISTICAL CONVERGENCE IN WEIGHTED SPACES

Let ρ denotes a continuous weight function with $\rho(x)\geq 1$, $x\in[0,\infty)$ and $\rho(x)\to\infty$ as $x\to\infty$. Let B_ρ be the weighted space of all functions f defined on the $\mathbb{R}_+$ satisfying the condition $|f(x)|\leq M_f\rho(x)$ with some constant M_f, depending only on f. By C_ρ, let us denote the subspace of all continuous functions belong to B_ρ. Also, let C_ρ^0 be the subspace of all functions $f \in C_\rho$ for which $\lim\limits_{|x|\to\infty} f(x)/\rho(x) = 0$. Endowed with the norm $\|f\|_\rho = \sup_{x\geq 0}\left(|f(x)|/\rho(x)\right)$ these spaces are Banach spaces. Note that the weighted Korovkin type theorem were proved by A.D. Gadjiev [10, 11]. Using A-statistical convergence, the weighted Korovkin type theorem was given in [6].

Let $\{L_{n,q}\}$ be the sequence of linear positive operators defined by (5). Then it is easily seen that $L_{n,q} : C_\rho \to B_\rho$.

Let $q = \{q_n\}$ be a sequence satisfying the following conditions

$$st_A - \lim_n q_n = 1 \quad \text{and} \quad st_A - \lim_n q_n^n = a, \quad (0 \leq a < 1)\,. \tag{12}$$

The condition (12) guaranties that $st_A - \lim_n \left([n]_q^{-1}\right) = 0$.

Now we are ready to prove our first result which is related to the A - statistical convergence the sequence of $\{L_{n,q_n}(f)\}$ to f.

Theorem 3.1. *Let $A = (a_{jn})$ be non-negative regular summability matrix, the sequence $q = \{q_n\}$ satisfies (12) with $q_n \in (0,1]$ for all $n \in \mathbb{N}$. Then for every $f \in C_\rho^0[0,\infty)$, $st_A - \lim\limits_n \|L_{n,q_n}(f) - f\|_\rho = 0$ where $\rho(x) = 1 + x^2$.*

Proof. From Lemma1, it is obvious that $st_A - \lim\limits_n \|L_{n,q}(e_0) - e_0\|_\rho = 0$. Using the (6), we get

$$\begin{aligned}
\frac{|L_{n,q_n}(e_1)(x) - e_1(x)|}{1+x^2} &= \frac{x}{1+x^2}\left|\frac{D_{q_n}\varphi_n\left([n]_{q_n}^{\beta-1}x\right)}{[n]_{q_n}\varphi_n([n]_{q_n}^{\beta-1}x)} - 1\right| \\
&\leq \|e_1\|_\rho \left|\frac{D_{q_n}\varphi_n\left([n]_{q_n}^{\beta-1}x\right)}{[n]_{q_n}\varphi_n([n]_{q_n}^{\beta-1}x)} - 1\right| \\
&\leq \left|\frac{D_{q_n}\varphi_n\left([n]_{q_n}^{\beta-1}x\right)}{[n]_{q_n}\varphi_n([n]_{q_n}^{\beta-1}x)} - 1\right| = B_{n,q_n}(\varphi_n, x)\,.
\end{aligned}$$

Now, for a given $\varepsilon > 0$, we define the sets $U = \left\{n : \|L_{n,q_n}(e_1) - e_1\|_\rho \geq \varepsilon\right\}$ and $U_1 = \{n : B_{n,q_n}(\varphi_n, x) \geq \varepsilon\}$. It is clear that $U \subset U_1$ and hence

$$\delta_A\left\{n \in \mathbb{N} : \|L_{n,q_n}(e_1) - e_1\|_\rho \geq \varepsilon\right\} \leq \delta_A\{n \in \mathbb{N} :\ B_{n,q_n}(\varphi_n, x) \geq \varepsilon\}\,.$$

From the condition (3) we get $st_A - \lim B_{n,q_n}(\varphi_n, x) = 0$. Therefore, it is clear that

$$\delta_A\{n \in \mathbb{N} :\ B_{n,q_n}(\varphi_n, x) \geq \varepsilon\} = 0$$

and hence we have

$$\delta_A\left\{n \in \mathbb{N} : \|L_{n,q_n}(e_1) - e_1\|_\rho \geq \varepsilon\right\} = 0,$$

which implies

$$st_A - \lim \left\| L_{n,q_n}(e_1) - e_1 \right\|_\rho = 0.$$

Similarly from (8) we can write

$$\left| L_{n,q_n}(e_2) - e_2 \right| = \left(\frac{D^2_{q_n} \varphi_n \left([n]^{\beta-1}_{q_n} x \right)}{q_n \ [n]^2_{q_n} \ \varphi_n([n]^{\beta-1}_{q_n} x)} - 1 \right) x^2$$
$$+ \frac{1}{[n]^\beta_{q_n}} \left(\frac{D_{q_n} \varphi_n \left([n]^{\beta-1}_{q_n} x \right)}{[n]_{q_n} \ \varphi_n([n]^{\beta-1}_{q_n} x)} - 1 \right) x + \frac{1}{[n]^\beta_{q_n}} x$$

and hence we get

$$\frac{\left| L_{n,q_n}(e_2) - e_2 \right|}{1+x^2} \le \|e_2\|_\rho \left| \frac{D^2_{q_n} \varphi_n \left([n]^{\beta-1}_{q_n} x \right)}{q_n \ [n]^2_{q_n} \ \varphi_n([n]^{\beta-1}_{q_n} x)} - 1 \right|$$
$$+ \frac{1}{[n]^\beta_{q_n}} \|e_1\|_\rho \left| \frac{D_{q_n} \varphi_n \left([n]^{\beta-1}_{q_n} x \right)}{[n]_{q_n} \ \varphi_n([n]^{\beta-1}_{q_n} x)} - 1 \right| + \frac{1}{[n]^\beta_{q_n}} \|e_1\|_\rho$$
$$\le B_{n,q_n}(\varphi_n, x) + \frac{1}{[n]^\beta_{q_n}} C_{n,q}(\varphi_n, x) + \frac{1}{[n]^\beta_{q_n}},$$

Now for given $\varepsilon > 0$, let us define the following sets

$$V = \left\{ n : \| L_{n,q_n}(e_2) - e_2 \|_\rho \ge \varepsilon \right\},$$
$$V_1 = \left\{ n : B_{n,q_n}(\varphi_n, x) \ge \varepsilon/3 \right\},$$
$$V_2 = \left\{ n : C_{n,q_n}(\varphi_n, x) \ge \varepsilon/3 \right\},$$
$$V_3 = \left\{ n : [n]^{-\beta}_{q_n} \ge \varepsilon/3 \right\}.$$

It is obviously that $V \subset V_1 \cup V_2 \cup V_3$. From the condition (12) we get

$$st_A - \lim \frac{1}{[n]^\beta_{q_n}} = st_A - \lim_{n\to\infty} \left((1-q_n) / (1-q_n^n) \right)^\beta = 0 \tag{13}$$

with $0 < \beta \le 2/3$. Hence by the condition (3) we have $st_A - \lim B_{n,q_n}(\varphi_n, x) = 0$ and $st_A - \lim C_{n,q_n}(\varphi_n, x) = 0$. Then we obtain $\delta_A(V_k) = 0$, $k = 1, 2, 3$. Since $\delta_A(V) \le \delta_A(V_1) + \delta_A(V_2) + \delta_A(V_3)$ we find that $st_A - \lim \| L_{n,q_n}(e_2) - e_2 \|_\rho = 0$.

Consequently we obtain that $st_A - \lim \| L_{n,q_n}(e_i) - e_i \|_\rho = 0, i = 0, 1, 2$ which completes the proof of the Theorem according to the weighted Korovkin type Theorem [12, 6, 10]. □

As a consequence, for all $n \in \mathbb{N}$, $x \ge 0$ and $0 < q_n < 1$, we have

$$st_A - \lim_n \| L_{n,q_n}((e_1 - e_0 x)^\nu) \|_\rho = 0, \nu = 1, 2 \tag{14}$$

where

$$L_{n,q_n}\left(\left(e_1-e_0x\right)\right)(x)=\left(\frac{D_{q_n}\varphi_n\left([n]_{q_n}^{\beta-1}x\right)}{[n]_{q_n}\varphi_n([n]_{q_n}^{\beta-1}x)}-1\right)x, \tag{15}$$

$$\begin{aligned}
&L_{n,q_n}\left(\left(e_1-e_0x\right)^2\right)(x)\\
&=\left(\frac{D_{q_n}^2\varphi_n\left([n]_{q_n}^{\beta-1}x\right)}{q_n\ [n]_{q_n}^2\ \varphi_n([n]_{q_n}^{\beta-1}x)}-2\frac{D_{q_n}\varphi_n\left([n]_{q_n}^{\beta-1}x\right)}{[n]_{q_n}\ \varphi_n([n]_{q_n}^{\beta-1}x)}+1\right)x^2\\
&\quad+\frac{1}{[n]_{q_n}^{\beta}}\left(\frac{D_{q_n}\varphi_n\left([n]_{q_n}^{\beta-1}x\right)}{[n]_{q_n}\ \varphi_n([n]_{q_n}^{\beta-1}x)}-1\right)x+\frac{1}{[n]_{q_n}^{\beta}}x.
\end{aligned} \tag{16}$$

Theorem 3.2. *Let $A=(a_{jn})$ be non-negative regular summability matrix, the sequence $q=\{q_n\}$ satisfies (12) with $q_n\in(0,1]$ for all $n\in\mathbb{N}$. If any function $f\in C_\rho$, satisfies the Lipschitz condition that is*

$$|f(t)-f(x)|\leq M\,|t-x|^{\alpha},0\leq\alpha<1,\ x,t\geq 0$$

then

$$st_A-\lim_n\sup_{x\geq 0}\frac{|L_{n,q_n}(f)(x)-f(x)|}{1+x^{\alpha}}=0$$

where M is a constant.

Proof. Since $L_{n,q_n}$is a linear positive operator and f satisfies the Lipschitz condition we can write,

$$\begin{aligned}
&|L_{n,q_n}(f)(x)-f(x)|\leq L_{n,q_n}(|f(t)-f(x)|)(x)\\
&\leq\frac{M}{\varphi_n([n]_{q_n}^{\beta-1}x)}\sum_{k=0}^{\infty}\left|\frac{[k]_q}{[n]_{q_n}^{\beta}\,q_n^{k-1}}-x\right|^{\alpha}\frac{([n]_{q_n}^{\beta-1}x)_n^k}{[k]_{q_n}!}D_{q_n}^k\varphi_n(0).
\end{aligned}$$

Applying the Holder inequality with $p=2/\alpha, s=2/(2-\alpha)$ and saying $B_{n,q_n,k}(\varphi_n;x):=\varphi_n^{-1}([n]_{q_n}^{\beta-1}x)\frac{([n]_{q_n}^{\beta-1}x)_n^k}{[k]_{q_n}!}D_{q_n}^k\varphi_n(0)$, from Lemma 1 we get

$$\begin{aligned}
|L_{n,q_n}(f)(x)-f(x)|&\leq M\left(\sum_{k=0}^{\infty}\left(\frac{[k]_{q_n}}{[n]_{q_n}^{\beta}\,q_n^{k-1}}-x\right)^2B_{n,q_n,k}(\varphi_n;x)\right)^{\alpha/2}\\
&\quad\times\left(\sum_{k=0}^{\infty}B_{n,q_n,k}(\varphi_n;x)\right)^{(2-\alpha)/2}\\
&=M\left(L_{n,q_n}\left(\left(e_1-e_0x\right)^2\right)(x)\right)^{\alpha/2}.
\end{aligned}$$

Taking account to (16) and using the conditions (3) and (12),similarly with the proof of the Theorem1, we obtain the desired result. □

Now, we concern with the order of approximation of a function $f \in C_\rho^0$ by the linear positive operator $L_{n,q}$. We will use the weighted modulus of continuity defined by

$$\Omega_m(f;\delta) = \sup_{x\in[0,\infty),\ |h|\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^m},$$

for each $f \in C_\rho^0,\ \rho(x) = 1 + x^m,\ x \in [0,\infty),\ m \in \mathbb{N}$.

The weighted modulus of continuity has the following properties (see [14]):

(i) $\lim_{\delta\to 0} \Omega_m(f;\delta) = 0$ for each $f \in C_\rho^0$

(ii) $\Omega_m(f;\lambda\delta) \leq (\lambda+1)\,\Omega_m(f;\delta)$ for each positive real number λ, $m \in \mathbb{N}$

(iii) $|f(t)-f(x)| \leq (1+(x+|t-x|)^m)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f;\delta)$ for every $x, t \in [0,\infty), m \in \mathbb{N}$.

Notice that, if f is not uniformly continuous on the interval $[0,\infty)$; then the usual first modulus of continuity $\omega(f;\delta)$ does not tend to zero, as $\delta \to 0$. It is seen that $\Omega_m(f;\delta) \to 0$,as $\delta \to 0$ for all $f \in C_\rho^0$ due to the property (i).

We now give second our main result. The following theorem is given an estimate for the approximation error with the operators $L_{n,q_n}(f)$, by means of $\Omega_1(f;\delta)$ with $\rho(x) = 1 + x$.

Theorem 3.3. *Let $\{q_n\}$ be a sequence satisfying the condition (12) with $q_n \in (0,1]$ for all $n \in \mathbb{N}$. Suppose that the condition*

$$\left(\frac{D_{q_n}^{\nu}\varphi_n([n]_{q_n}^{\beta-1}x)}{q_n^{\nu-1}[n]_{q_n}^{\nu}\varphi_n([n]_{q_n}^{\beta-1}x)}-1\right) = O\left(1/[n]_{q_n}^{\beta}\right)$$

holds instead of (3). If $f \in C_\rho^0$ with $\rho(x) = 1 + x$ then the inequality

$$(17)\qquad \left\|L_{n,q_n}(f)(x) - f(x)\right\|_{\rho_2} \leq C\ \Omega_1\left(f; 1/\sqrt{[n]_{q_n}^{\beta}}\right)\left(1 + 1/[n]_{q_n}^{\beta}\right)$$

holds where $\rho_2(x) = 1 + x^2$ and C is a constant independent of f and n.

Proof. Considering the definition of $\Omega_1(f;\delta)$ and by using the property (iii) of $\Omega_1(f;\delta)$ we can write

$$\begin{aligned}|f(t)-f(x)| &\leq (1+x+|t-x|)\left(\frac{|t-x|}{\delta}+1\right)\Omega_1(f;\delta)\\ &\leq (1+2x+t)\left(\frac{|t-x|}{\delta}+1\right)\Omega_1(f;\delta).\end{aligned}$$

Since L_{n,q_n} is a linear and positive operator we get

$$\begin{aligned}|L_{n,q_n}(f)(x)-f(x)| &\leq L_{n,q_n}(|f(t)-f(x)|)(x)\\ &\leq \Omega_1(f;\delta)\Big[L_{n,q_n}(1+2x+t)(x)\\ &\quad + L_{n,q_n}\left((1+2x+t)\frac{|t-x|}{\delta}\right)(x)\Big].\end{aligned} \tag{18}$$

To estimate the first term, considering (6) and (7), we can write

$$\begin{aligned}&\left(L_{n,q_n}((1+2x+t))(x)\right)\\ &= (1+2x)L_{n,q_n}(e_0)(x)+L_{n,q_n}(e_1)(x)\\ &= (1+3x)+\left(\frac{1}{[n]_{q_n}^{\beta}}\frac{D_{q_n}\varphi_n\left([n]_{q_n}^{\beta-1}x\right)}{[n]_{q_n}\varphi_n([n]_{q_n}^{\beta-1}x)}-1\right)x\\ &\leq 3(1+x)\left[1+O\left(1/[n]_{q_n}^{\beta}\right)\right].\end{aligned} \tag{19}$$

Applying the Cauchy-Schwarz inequality to the second term in (18), since L_{n,q_n} is a linear and positive, we get

$$\begin{aligned}&L_{n,q_n}\left((1+2x+t)\tfrac{|t-x|}{\delta}\right)(x)\\ &\leq \left(L_{n,q_n}\left((1+2x+t)^2\right)(x)\right)^{1/2}\times\left(L_{n,q_n}\left(\tfrac{(t-x)^2}{\delta^2}\right)(x)\right)^{1/2}.\end{aligned} \tag{20}$$

We now estimate the first term. By using (6),(7) and (8) and by simple calculations ,we get

$$\left(L_{n,q_n}\left((1+2x+t)^2\right)(x)\right)^{1/2}\leq 4(1+x)\ \left[1+O\left(\frac{1}{[n]_{q_n}^{\beta}}\right)\right]^{1/2}. \tag{21}$$

Taking into account (16), if we estimate the second term then we get

$$\begin{aligned}\left(L_{n,q_n}\left(\frac{|t-x|^2}{\delta^2}\right)(x)\right)^{1/2} &= \frac{1}{\delta}\left(L_{n,q_n}\left((e_1-e_0x)^2\right)(x)\right)^{1/2}\\ &= \frac{1}{\delta}\left(O\left(\frac{1}{[n]_{q_n}^{\beta}}\right)(x^2+x)\right)^{1/2}\\ &\leq \frac{1}{\delta}\left(O\left(\frac{1}{[n]_{q_n}^{\beta}}\right)(1+x)^2\right)^{1/2}\\ &\leq \frac{1}{\delta}(1+x)\sqrt{\frac{C_1}{[n]_{q_n}^{\beta}}}\end{aligned} \tag{22}$$

with C_1 is a constant independent of n. Combining (19), (20), (21) and (22) with (18) we have

$$\begin{aligned}&\left|L_{n,q_n}(f)(x)-f(x)\right|\\ &\leq \Omega_1(f;\delta)\times C_2(1+x)^2\left[1+O\left(1/[n]_{q_n}^{\beta}\right)\right]\left(1+\frac{1}{\delta}\sqrt{\frac{1}{[n]_{q_n}^{\beta}}}\right),\end{aligned}$$

where $C_2=\max\{3,4C_1\}$. Taking $\delta:=\delta_n=\left(1/[n]_{q_n}^{\beta}\right)^{1/2}$ we obtain

$$\begin{aligned}\left|L_{n,q_n}(f)(x)-f(x)\right| &\leq 2C_2\Omega_1(f;\delta_n)\left(1+x^2\right)\left[C_3+O\left(1/[n]_{q_n}^{\beta}\right)\right]\\ &\leq C\Omega_1(f;\delta_n)\left(1+x^2\right)\left[1+O\left(1/[n]_{q_n}^{\beta}\right)\right],\end{aligned}$$

with $C_3=\sup_{x\geq 0}\frac{(1+x)^2}{1+x^2}, C=\max\{2C_2,C_3,C_3C_1\}$ which gives that the (17).

We notice that, from (13), it is clear that $st_A-\lim_n\delta_n\to 0$ as $n\to\infty$. Therefore $st_A-\lim_n\Omega_1(f;\delta_n)\to 0$ due to the property (i) of $\Omega_1(f;\delta)$. Consequently, order of A-statistical convergence of the sequence of $\{L_{n,q_n}(f)\}$ to f is $\left(1/[n]_{q_n}^{\beta}\right)^{1/2}$ in the ρ_2-norm. □

4. LOCAL APPROXIMATION

Theorem 4.1. *Let $\{q_n\}$ be a sequence satisfying the condition (12) with $q_n\in(0,1]$ for all $n\in\mathbb{N}$. We have*

1) *For any $f\in C_\rho$ we have*

$$\left|L_{n,q_n}(f)(x)-f(x)\right|\leq 2\omega(f;\sqrt{\delta_{n,x}})$$

where $\omega(f,\delta)$ is the usual first modulus of continuity of f and

$$\delta_{n,x}=L_{n,q_n}\left((e_1-e_0x)^2\right)(x) \tag{23}$$

and $st_A-\lim_n\delta_{n,x}=0$, for all fixed $x\in[0,\infty)$.

2) *If $f\in C_\rho$ satisfies the Lipschitz condition then $\left|L_{n,q_n}(f)(x)-f(x)\right|\leq M\delta_{n,x}^{\alpha/2},\ 0\leq\alpha<1$.*

Proof. 1) Using the linearity and positivity of the operator L_{n,q_n} and the known properties of $\omega(f,\delta)$ and applying Cauchy -Schwarz inequality we obtain

$$\begin{aligned}&\left|L_{n,q_n}(f)(x)-f(x)\right|\leq L_{n,q_n}(|f(t)-f(x)|)(x)\\ &\leq\omega(f;\delta)\left[L_{n,q_n}(e_0)(x)+\frac{1}{\delta}\left(L_{n,q_n}(e_1-e_0x)^2(x)\right)^{1/2}\right].\end{aligned}$$

By choosing $\delta=\sqrt{\delta_{n,x}}$ as in (23), we reach the desired result. Notice that, taking into account (14), we get $st_A-\lim_n\delta_{n,x}=0$ for all fixed x.

Hence we have $st_A - \lim_n \omega(f;\delta_{n,x}) = 0$. This gives the pointwise rate of A-statistical convergence of the operator $L_{n,q_n}(f)$ to the function f.

2) Considering the proof of Theorem 2 and formula (23) we obtain that

$$\begin{aligned} |L_{n,q_n}(f)(x) - f(x)| &\leq M\left(L_{n,q_n}\left((e_1 - e_0x)^2\right)(x)\right)^{\alpha/2} \\ &= M\delta_{n,x}^{\alpha/2}. \end{aligned}$$ □

Now we give the rate of $A-$statistical convergence for the operators $L_{n;q_n}(f)$ by using the Peetre's $K-$functional in the space $C_B^2[0,\infty)$.

Let $C_B[0,\infty)$ be the space of all real valued uniformly continuous and bounded functions f on the interval $[0,\infty)$ with the norm

$$\|f\|_{C_B} = \sup_{x\in[0,\infty)} |f(x)|.$$

The Peetre's K-functional of function $f \in C_B[0,\infty)$ is defined by

$$\mathcal{K}(f;\delta) = \inf_{g\in C_B^2}\left\{\|f-g\|_{C_B} + \delta\|g\|_{C_B^2}\right\}$$

where $\delta > 0$ and $C_B^2[0,\infty) = \{f \in C_B : f', f'' \in C_B[0,\infty)\}$ endowed with the norm

$$\|f\|_{C_B^2} = \|f\|_{C_B} + \left\|f^{'}\right\|_{C_B} + \left\|f^{''}\right\|_{C_B}. \tag{24}$$

Theorem 4.2. *For each $f \in C_B^2[0,\infty)$ we have*

$$st_A - \lim_n \|L_{n,q_n}f - f\|_{C_B} = 0.$$

Proof. Applying the Taylor expansion to the function $f \in C_B^2[0,\infty)$, we can write

$$\begin{aligned} L_{n,q_n}(f)(x) - f(x) &= f''(x)L_{n,q_n}((e_1 - e_0x))(x) \\ &\quad + \frac{1}{2}f''(\zeta)L_{n,q_n}\left((e_1 - e_0x)^2\right)(x), \quad \zeta \in (t,x), \end{aligned}$$

where $L_{n,q_n}((e_1 - e_0x))(x), L_{n,q_n}\left((e_1 - e_0x)^2\right)(x)$ are given by (15) and (16) respectively.

Hence

$$\begin{aligned} \|L_{n,q_n}(f) - f\|_{C_B} &\leq \left\|f'\right\|_{C_B}\|L_{n,q_n}((e_1 - e_0x))\|_{C[0,A]} \\ &\quad + \left\|f''\right\|_{C_B}\left\|L_{n,q_n}\left((e_1 - e_0x)^2\right)\right\|_{C[0,\alpha]}. \end{aligned} \tag{25}$$

Now for given $\varepsilon > 0$, let us define $U = \left\{n \in \mathbb{N} : \|L_{n,q_n}(f) - f\|_{C[0,\alpha]} \geq \varepsilon\right\}$,

$$U_1 = \left\{n \in \mathbb{N} : \left\|f'\right\|_{C_B}\|L_{n,q_n}((e_1 - e_0x))\|_{C[0,\alpha]} \geq \varepsilon/2\right\},$$

$$U_2=\left\{n\in\mathbb{N}:\|f''\|_{C_B}\left\|L_{n,q_n}\left((e_1-e_0x)^2\right)\right\|_{C[0,\alpha]}\geq\varepsilon/2\right\}.$$

It is obvious that $U\subset U_1\cup U_2$ and hence $\delta_A U\leq\delta_A U_1+\delta_A U_2$. By using (14) we get $st_A-\lim_n\|L_{n,q_n}((e_1-e_0x)^\nu)\|_{C[0,\alpha]}=0,\nu=1,2$ with $[0,\alpha]\subset[0,\infty)$ and $\|,\|_{C[0,\alpha]}$ is maximum norm. Therefore we obtain $\delta_A U_1=0,\ \delta_A U_2=0$ so $\delta_A U=0$ which completes the proof. □

Theorem 4.3. *For each $f\in C_B[0,\infty)$*

$$\|L_{n,q_n}(f)-f\|_{C_B}\leq\mathcal{K}(f;\delta_{n,x})$$

where $\{\mathcal{K}(f;\delta_{n,x})\}$is the sequence of Peetre's K-functional and

$$\delta_{n,x}=\|L_{n,q_n}(e_1-e_0x)\|_{C[0,\alpha]}+\left\|L_n(e_1-e_0x)^2\right\|_{C[0,\alpha]}$$

and $st_A-\lim_n\delta_{n,x}=0$ for each fixed $x\in[0,\infty)$.

Proof. For each $g\in C_B^2[0,\infty)$,by using (24) and (25), we get

$$\begin{aligned}\|L_{n,q}g-g\|_{C_B^2}&\leq\Big(\|L_{n,q_n}(e_1-e_0x)\|_{C[0,\alpha]}\\&\qquad+\left\|L_{n,q_n}(e_1-e_0x)^2\right\|_{C[0,\alpha]}\Big)\|g\|_{C_B^2}\\&=\delta_{n,x}\|g\|_{C_B^2}\ \text{say.}\end{aligned}$$

For each $f\in C_B[0,\infty)$ and $g\in C_B^2[0,\infty)$, we obtain

$$\begin{aligned}\|L_{n,q_n}f-f\|_{C_B^2}&\leq\|L_{n,q_n}f-L_{n,q_n}g\|_{C_B}+\|L_{n,q_n}g-g\|_{C_B^2}+\|g-f\|_{C_B}\\&\leq2\|g-f\|_{C_B}+\|L_{n,q_n}g-g\|_{C_B^2}\\&\leq2\|g-f\|_{C_B}+\delta_{n,x}\|g\|_{C_B^2}\\&\leq2\left(\|g-f\|_{C_B}+\delta_{n,x}\|g\|_{C_B^2}\right).\end{aligned}$$

Taking the infimum on the right hand side over all $g\in C_B^2[0,\infty)$ we get

$$\|L_{n,q_n}f-f\|_{C_B^2}\leq\mathcal{K}(f;\delta_{n,x}).$$

By (14), we get $st_A-\lim_n\delta_{n,x}=0$ so $st_A-\lim_n\mathcal{K}(f;\delta_{n,x})=0$. Therefore we obtain the rate of $A-$statistical convergence of the sequence of the operators $L_{n,q_n}(f)$ to f in the space $C_B[0,\infty)$. □

5. Concluding Remarks

Some particular cases of the operators $L_{n,q}$ are defined as follows:

a) If we take $\varphi_n(x) = (1+x)_q^n$ then we obtain q-Balazs-Szabados operators which are studied by O. Dogru [5]. In [5], the function f has been taken as $f\left([k]_q / [n]_q^\beta\right)$ instead of $f\left([k]_q / q^{k-1}[n]_q^\beta\right)$ which is a natural generalization of q-Balazs-Szabados operators.

b) Taking into account the q-analogues of the exponential function given by $E_q\left([n]_q x\right) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{\left([n]_q^\beta x\right)^k}{[k]_q!}$ and $e_q\left([n]_q x\right) = \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!}$ choosing $\varphi_n(x) = E_q\left(-[n]_q x\right)$ or $\varphi_n(x) = e_q\left(-[n]_q x\right)$, with $\beta = 1$, we obtain q-Szasz-Mirakjan operators studied in different spaces in [1] and [20], respectively.

c) Taking $\varphi_n(x) = \left(1+q^{n-1}x\right)_q^{-n}$ we obtain the q-analogue of classical Baskakov operators studied in [2].

Consequently the $A-$statistical approximation properties are valid in large spectrum of the operators (5).

If we take $A = I$, identity matrix, we have the ordinary rate of convergence for the operators (5) (see, [14, 16]).

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