

# On a New Subclass of Harmonic Univalent Functions Defined by Multiplier Transformation

SAURABH PORWAL

ABSTRACT. The purpose of the present paper is to introduce a new subclass of harmonic univalent functions by using Multiplier transformation. Coefficient estimates, distortion bounds, extreme points, convolution condition and convex combination for functions belonging to this class are determined. The results obtained for the class reduce to the corresponding several known results are briefly indicated.

## 1. INTRODUCTION

A continuous complex-valued function  $f = u + iv$  defined in a simply-connected domain  $D$  is said to be harmonic in  $D$  if both  $u$  and  $v$  are real harmonic in  $D$ . In any simply-connected domain  $D$ , we can write  $f = h + \bar{g}$ , where  $h$  and  $g$  are analytic in  $D$ . We call  $h$  the analytic part and  $g$  the co-analytic part of  $f$ . A necessary and sufficient condition for  $f$  to be locally univalent and sense-preserving in  $D$  is that  $|h'(z)| > |g'(z)|$ ,  $z \in D$ . See Clunie and Sheil-Small [2], for more basic results on harmonic functions one may refer to the following standard text book by Duren [8], (see also [1] and [11]).

Denote by  $S_H$  the class of functions  $f = h + \bar{g}$  which are harmonic univalent and sense-preserving in the open unit disk  $U = \{z : |z| < 1\}$  for which  $f(0) = f_z(0) - 1 = 0$ . Then for  $f = h + \bar{g} \in S_H$  we may express the analytic functions  $h$  and  $g$  as

$$(1) \quad h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1.$$

Note that  $S_H$  reduces to the class  $S$  of normalized analytic univalent functions if the co-analytic part of its member is zero.

For this class the function  $f(z)$  may be expressed as

$$(2) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

---

2010 *Mathematics Subject Classification*. Primary: 30C45; Secondary: 30C50.

*Key words and phrases*. Harmonic, Univalent function, Multiplier transformation.

Cho and Srivastava [3], (see also [4]), introduced the operator  $I_\lambda^n : A \rightarrow A$  defined as

$$(3) \quad I_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n a_k z^k,$$

where  $A$  denote the class of functions of the form (2) which are analytic in the open unit disc  $U$ .

For  $\lambda = 1$ , the operator  $I_\lambda^n \equiv I^n$  was studied by Uralegaddi and Somanatha [19] and for  $\lambda = 0$  the operator  $I_\lambda^n$  reduce to well-known Salagean operator introduced by Salagean [18].

Motivated with the definition of modified Salagean operator introduced by Jahangiri *et al.* [9], we define the modified multiplier transformation for  $f = h + \bar{g}$  given by (1) as

$$(4) \quad I_\lambda^n f(z) = I_\lambda^n h(z) + (-1)^n \overline{I_\lambda^n g(z)}, \quad (n \in N_0 = N \cup \{0\}),$$

where

$$I_\lambda^n h(z) = z + \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n a_k z^k,$$

$$I_\lambda^n g(z) = \sum_{k=1}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n b_k z^k.$$

Now, for  $1 < \beta \leq \frac{4}{3}$ ,  $-1 < \lambda \leq 1$ ,  $n \in N_0$  and  $z \in U$ , suppose that  $S_H(n, \lambda, \beta)$  denote the family of harmonic functions  $f$  of the form (1) such that

$$(5) \quad \Re \left\{ \frac{I_\lambda^{n+1} f(z)}{I_\lambda^n f(z)} \right\} < \beta,$$

where  $I_\lambda^n f$  is defined by (4).

Further, let the subclass  $VS_H(n, \lambda, \beta)$  consisting of harmonic functions  $f_n = h + \bar{g}_n$  in  $S_H(n, \lambda, \beta)$  so that  $h$  and  $g_n$  are of the form

$$(6) \quad h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k, \quad g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1.$$

In 1994 Uralegaddi *et al.* [20] introduced an analogous subclass of starlike, convex and close-to-convex functions with positive coefficients and opened up a new and interesting direction of research in Geometric Function Theory. In fact, they considered the functions of the form (2) where the coefficients are positive. Motivated by the work of Uralegaddi *et al.* [20], several researchers (e.g. Dixit and Chandra [5], Dixit *et al.* [7], Porwal and Dixit [15], Porwal *et al.* [17]) introduce new subclasses of analytic functions with positive coefficients. Recently, analogues to these results, Dixit and Porwal [6] introduced the class  $R_H(\beta)$  of harmonic univalent functions with positive coefficients and opened up a new direction of research in the theory of

harmonic univalent functions. After the appearance of this paper several researchers (e.g. see Pathak *et al.* [10], Porwal and Aouf [12], Porwal *et al.* [16]) generalized the result of [6] by using certain operator. Very recently Porwal and Dixit [13] (see also [14]) investigated new subclasses of harmonic starlike and convex functions. In the present paper, analogues to these results, we study the subclass of harmonic univalent functions defined by Multiplier transformation in which  $h$  has positive coefficients.

Assigning specific values to  $n$ ,  $\lambda$  and  $\beta$  in the subclass  $S_H(n, \lambda, \beta)$ , we obtain the following known subclasses studied earlier by various researchers.

- (1) If we put  $\lambda = 0$  then it reduces to the class  $S_H(n, \beta)$  studied by Porwal and Dixit [14].
- (2) If we put  $n = 0, \lambda = 0$  then it reduces to the class  $L_H(\beta)$  studied by Porwal and Dixit [13].
- (3) If we put  $n = 1, \lambda = 0$  then it reduces to the class  $M_H(\beta)$  studied by Porwal and Dixit [13].
- (4) If we put  $n = 0$  and  $n = 1$  with  $\lambda = 0, g \equiv 0$ , then it reduces to the classes  $L(\beta)$  and  $M(\beta)$  studied by Uralegaddi *et al.* [20].
- (5) For  $g \equiv 0$  then it reduces to the class  $L(n, j, \beta)$  for  $j = 1$  studied by Dixit and Chandra [5].

In the present paper, we extend the above results to the families  $S_H(n, \lambda, \beta)$  and  $VS_H(n, \lambda, \beta)$ . We also obtain extreme points, distortion bounds, convolution conditions and convex combinations of functions belonging to the class  $VS_H(n, \lambda, \beta)$ .

## 2. MAIN RESULTS

We first prove a sufficient coefficient condition for function in  $S_H(n, \lambda, \beta)$ .

**Theorem 2.1.** *Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1). Furthermore, let*

$$(7) \quad \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)} \right) |a_k| + \sum_{k=1}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)} \right) |b_k| \leq 1,$$

where  $n \in \mathbb{N}_0$ ,  $-1 < \lambda \leq 1$ ,  $1 < \beta \leq \frac{4}{3}$ , then  $f$  is sense-preserving, harmonic univalent in  $U$  and  $f \in S_H(n, \lambda, \beta)$ .

*Proof.* If  $z_1 \neq z_2$ , then

$$\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right|$$

$$\begin{aligned}
&= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)} \right| \\
&> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \\
&\geq 1 - \frac{\sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)}\right) |b_k|}{1 - \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)}\right) |a_k|} \\
&\geq 0,
\end{aligned}$$

which proves univalence.

Note that  $f$  is sense-preserving in  $U$ . This is because

$$\begin{aligned}
|h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1 - \sum_{k=2}^{\infty} k|a_k| \\
&\geq 1 - \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)}\right) |a_k| \\
&\geq \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)}\right) |b_k| \\
&\geq \sum_{k=1}^{\infty} k|b_k| > \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.
\end{aligned}$$

To prove that  $f \in S_H(n, \lambda, \beta)$ . Using the fact that  $\Re \omega < \beta$ , if and only if,  $|\omega - 1| < |\omega + 1 - 2\beta|$ , it suffies to show that

$$\left| \frac{\frac{I_{\lambda}^{n+1}f(z)}{I_{\lambda}^n f(z)} - 1}{\frac{I_{\lambda}^{n+1}f(z)}{I_{\lambda}^n f(z)} + 1 - 2\beta} \right| < 1, \quad z \in U.$$

We have

$$\left| \frac{\frac{I_{\lambda}^{n+1}f(z)}{I_{\lambda}^n f(z)} - 1}{\frac{I_{\lambda}^{n+1}f(z)}{I_{\lambda}^n f(z)} + 1 - 2\beta} \right| =$$

$$\begin{aligned}
&= \frac{\left( \sum_{k=2}^{\infty} \left( \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} - \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) a_k z^k \right.}{\left( 2(1-\beta)z + \sum_{k=2}^{\infty} \left( \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} + (1-2\beta) \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) a_k z^k \right)} \\
&\quad \left. + (-1)^n \sum_{k=1}^{\infty} \left( - \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} - \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) \overline{b_k z^k} \right) \\
&\leq \frac{\left( \sum_{k=2}^{\infty} \left( \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} - \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) |a_k| |z|^{k-1} \right.}{\left( 2(\beta-1) - \sum_{k=2}^{\infty} \left( \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} + (1-2\beta) \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) |a_k| |z|^{k-1} \right)} \\
&\quad \left. + \sum_{k=1}^{\infty} \left( \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} + \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) |b_k| |z|^{k-1} \right) \\
&< \frac{\left( \sum_{k=2}^{\infty} \left( \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} - \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) |a_k| \right.}{\left( 2(\beta-1) - \sum_{k=2}^{\infty} \left( \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} + (1-2\beta) \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) |a_k| \right)} \\
&\quad \left. + \sum_{k=1}^{\infty} \left( \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} + \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) |b_k| \right)
\end{aligned}$$

which is bounded above by 1 by using (7) and so the proof is complete.

The harmonic univalent functions

$$(8) \quad f(z) = z + \sum_{k=2}^{\infty} \frac{\beta - 1}{\left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)}\right)} x_k z^k \\ + \sum_{k=1}^{\infty} \frac{\beta - 1}{\left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)}\right)} \overline{y_k z^k},$$

where  $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$  show that the coefficient bound given by (7) is sharp.  $\square$

In the following theorem, it is shown that the condition (7) is also necessary for functions  $f_n = h + \overline{g}_n$  where  $h$  and  $g_n$  are of the form (6).

**Theorem 2.2.** *Let  $f_n = h + \overline{g}_n$  be given by (6).*

*Then  $f_n \in VS_H(n, \lambda, \beta)$ , if and only if*

$$(9) \quad \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)}\right) |a_k| + \\ \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)}\right) |b_k| \leq 1.$$

*Proof.* Since  $VS_H(n, \lambda, \beta) \subset S_H(n, \lambda, \beta)$ , we only need to prove the “only if” part of the theorem. To this end, for functions  $f_n$  of the form (6), we notice that the condition

$$\Re \left\{ \frac{I_{\lambda}^{n+1} f_n(z)}{I_{\lambda}^n f_n(z)} \right\} < \beta$$

is equivalent to

$$(10) \quad \Re \left\{ \frac{\left( (\beta-1)z - \sum_{k=2}^{\infty} \left( \left(\frac{k+\lambda}{1+\lambda}\right)^{n+1} - \beta \left(\frac{k+\lambda}{1+\lambda}\right)^n \right) |a_k| z^k \right) - (-1)^{2n} \sum_{k=1}^{\infty} \left( \left(\frac{k+\lambda}{1+\lambda}\right)^{n+1} + \beta \left(\frac{k+\lambda}{1+\lambda}\right)^n \right) |b_k| \overline{z}^k}{z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n |a_k| z^k + (-1)^{2n-1} \sum_{k=1}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^n |b_k| \overline{z}^k} \right\} > 0.$$

The above required condition (10) must hold for all values of  $z$  in  $U$ . Upon choosing the values of  $z$  on the positive real axis where  $0 \leq z = r < 1$ , we must have

$$(11) \quad \frac{\left( (\beta - 1) - \sum_{k=2}^{\infty} \left( \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} - \beta \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) |a_k| r^{k-1} \right) - \sum_{k=1}^{\infty} \left( \left( \frac{k+\lambda}{1+\lambda} \right)^{n+1} + \beta \left( \frac{k+\lambda}{1+\lambda} \right)^n \right) |b_k| r^{k-1}}{1 - \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n |a_k| r^{k-1} - \sum_{k=1}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n |b_k| r^{k-1}} \geq 0.$$

If the condition (9) does not hold, then the numerator in (11) is negative for  $r$  sufficiently close to 1. Hence there exists  $z_0 = r_0$  in  $(0,1)$  for which the quotient in (11) is negative. This contradicts the required condition for  $f_n \in VS_H(n, \lambda, \beta)$  and so the proof is complete.  $\square$

Next, we determine the extreme points of closed convex hulls of  $VS_H(n, \lambda, \beta)$  denoted by  $\text{clco } VS_H(n, \lambda, \beta)$ .

**Theorem 2.3.** *Let  $f_n$  be given by (6). Then  $f_n \in VS_H(n, \lambda, \beta)$ , if and only if*

$$(12) \quad f_n(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_{n_k}(z)),$$

where

$$\begin{aligned} h_1(z) &= z, \\ h_k(z) &= z + \frac{\beta - 1}{\left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k-\beta+\lambda(1-\beta)}{1+\lambda} \right)} z^k, \quad (k = 2, 3, \dots), \\ g_{n_k}(z) &= z + (-1)^{n-1} \frac{\beta - 1}{\left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k+\beta+\lambda(1+\beta)}{1+\lambda} \right)} \bar{z}^k, \quad (k = 1, 2, \dots), \end{aligned}$$

and

$$x_k \geq 0, \quad y_k \geq 0, \quad x_1 = 1 - \left( \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k \right) \geq 0.$$

In particular, the extreme points of  $VS_H(n, \lambda, \beta)$  are  $\{h_k\}$  and  $\{g_{n_k}\}$ .

*Proof.* For functions  $f$  of form (12), we have

$$\begin{aligned} f_n(z) &= \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_{n_k}(z)) = z + \sum_{k=2}^{\infty} \frac{\beta - 1}{\left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k-\beta+\lambda(1-\beta)}{1+\lambda} \right)} x_k z^k \\ &\quad + (-1)^{n-1} \sum_{k=1}^{\infty} \frac{\beta - 1}{\left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k+\beta+\lambda(1+\beta)}{1+\lambda} \right)} y_k \bar{z}^k. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{\left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k-\beta+\lambda(1-\beta)}{(1+\lambda)}\right)}{\beta-1} \left( \frac{\beta-1}{\left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k-\beta+\lambda(1-\beta)}{(1+\lambda)}\right)} x_k \right) \\ & + \sum_{k=1}^{\infty} \frac{\left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k+\beta+\lambda(1+\beta)}{(1+\lambda)}\right)}{\beta-1} \left( \frac{\beta-1}{\left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k+\beta+\lambda(1+\beta)}{(1+\lambda)}\right)} y_k \right) \\ & = \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1 \leq 1, \end{aligned}$$

and so  $f_n \in VS_H(n, \lambda, \beta)$ .

Conversely, suppose that  $f_n \in \text{clco } VS_H(n, \lambda, \beta)$ , then

$$\begin{aligned} |a_k| & \leq \frac{\beta-1}{\left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k-\beta+\lambda(1-\beta)}{(1+\lambda)}\right)}, \\ |b_k| & \leq \frac{\beta-1}{\left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k+\beta+\lambda(1+\beta)}{(1+\lambda)}\right)}. \end{aligned}$$

Set

$$\begin{aligned} x_k & = \frac{\left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k-\beta+\lambda(1-\beta)}{(1+\lambda)}\right)}{\beta-1} |a_k|, \quad (k = 2, 3, \dots), \\ y_k & = \frac{\left(\frac{k+\lambda}{1+\lambda}\right)^n \left(\frac{k+\beta+\lambda(1+\beta)}{(1+\lambda)}\right)}{\beta-1} |b_k|, \quad (k = 1, 2, \dots). \end{aligned}$$

Then note that by Theorem 2.2,  $0 \leq x_k \leq 1$ , ( $k = 2, 3, \dots$ ) and  $0 \leq y_k \leq 1$ , ( $k = 1, 2, \dots$ ). We define  $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$  and note that, by Theorem 2.2,  $x_1 \geq 0$ . Consequently, we obtain  $f_n(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_{n_k}(z))$  as required.  $\square$

The following theorem gives the distortion bounds for functions in  $VS_H(n, \lambda, \beta)$  which yields a covering result for the class.

**Theorem 2.4.** *Let  $f_n \in VS_H(n, \lambda, \beta)$ . Then for  $|z| = r < 1$ , we have*

$$\begin{aligned} |f_n(z)| & \leq (1 + |b_1|)r + \left(\frac{1+\lambda}{2+\lambda}\right)^n \left( \frac{(\beta-1)(\lambda+1)}{2-\beta+\lambda(1-\beta)} - \frac{(\beta+1)(\lambda+1)}{2-\beta+\lambda(1-\beta)} |b_1| \right) r^2, \\ |f_n(z)| & \geq (1 - |b_1|)r - \left(\frac{1+\lambda}{2+\lambda}\right)^n \left( \frac{(\beta-1)(\lambda+1)}{2-\beta+\lambda(1-\beta)} - \frac{(\beta+1)(\lambda+1)}{2-\beta+\lambda(1-\beta)} |b_1| \right) r^2. \end{aligned}$$



*Proof.* We only prove the right hand inequality. The proof for left hand inequality is similar and will be omitted. Let  $f_n(z) \in VS_H(n, \lambda, \beta)$ . Taking the absolute value of  $f$ , we have

$$\begin{aligned}
|f_n(z)| &\leq (1 + |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^k \\
&\leq (1 + |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^2 \\
&= (1 + |b_1|)r + \left(\frac{1+\lambda}{2+\lambda}\right)^n \frac{(\beta-1)(\lambda+1)}{2-\beta+\lambda(1-\beta)} \\
&\quad \cdot \sum_{k=2}^{\infty} \left( \left(\frac{2+\lambda}{1+\lambda}\right)^n \frac{2-\beta+\lambda(1-\beta)}{(\beta-1)(\lambda+1)} |a_k| + \left(\frac{2+\lambda}{1+\lambda}\right)^n \frac{2-\beta+\lambda(1-\beta)}{(\beta-1)(\lambda+1)} |b_k| \right) r^2 \\
&\leq (1 + |b_1|)r + \left(\frac{1+\lambda}{2+\lambda}\right)^n \frac{(\beta-1)(\lambda+1)}{2-\beta+\lambda(1-\beta)} \\
&\quad \cdot \sum_{k=2}^{\infty} \left( \left(\frac{k+\lambda}{1+\lambda}\right)^n \frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(\lambda+1)} |a_k| + \left(\frac{k+\lambda}{1+\lambda}\right)^n \frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(\lambda+1)} |b_k| \right) r^2 \\
&\leq (1 + |b_1|)r + \left(\frac{1+\lambda}{2+\lambda}\right)^n \frac{(\beta-1)(\lambda+1)}{2-\beta+\lambda(1-\beta)} \left(1 - \frac{\beta+1}{\beta-1} |b_1|\right) r^2 \\
&= (1 + |b_1|)r + \left(\frac{1+\lambda}{2+\lambda}\right)^n \left( \frac{(\beta-1)(\lambda+1)}{2-\beta+\lambda(1-\beta)} - \frac{(\beta+1)(\lambda+1)}{2-\beta+\lambda(1-\beta)} |b_1| \right) r^2. \quad \square
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 2.4.

**Corollary 2.1.** *Let  $f_n$  of the form (6) be so that  $f_n \in VS_H(n, \lambda, \beta)$ . Then*

$$\left\{ \omega : |\omega| < \frac{(2+\lambda)^n (2-\beta+\lambda(1-\beta)) - (1+\lambda)^{n+1} (\beta-1)}{(2+\lambda)^n (2-\beta+\lambda(1-\beta))} \right. \\
\left. - \frac{(2+\lambda)^n (2-\beta+\lambda(1-\beta)) - (1+\lambda)^{n+1} (\beta+1)}{(2+\lambda)^n (2-\beta+\lambda(1-\beta))} |b_1| \right\} \subset f_n(U).$$

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$\begin{aligned}
f_n(z) &= z + \sum_{k=2}^{\infty} |a_k| z^k + (-1)^{n-1} \sum_{k=1}^{\infty} |b_k| \bar{z}^k, \\
F_n(z) &= z + \sum_{k=2}^{\infty} |A_k| z^k + (-1)^{n-1} \sum_{k=1}^{\infty} |B_k| \bar{z}^k.
\end{aligned}$$

We define their convolution

$$(13) \quad \begin{aligned} (f_n * F_n)(z) &= f_n(z) * F_n(z) \\ &= z + \sum_{k=2}^{\infty} |a_k A_k| z^k + (-1)^{n-1} \sum_{k=1}^{\infty} |b_k B_k| \bar{z}^k. \end{aligned}$$

Using this definition, we show that the class  $VS_H(n, \lambda, \beta)$  is closed under convolution.

**Theorem 2.5.** For  $1 < \beta \leq \alpha \leq \frac{4}{3}$ , let  $f_n(z) \in VS_H(n, \lambda, \beta)$  and  $F_n(z) \in VS_H(n, \lambda, \alpha)$ . Then  $(f_n * F_n)(z) \in VS_H(n, \lambda, \beta) \subseteq VS_H(n, \lambda, \alpha)$ .

*Proof.* Let

$$f_n(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + (-1)^{n-1} \sum_{k=1}^{\infty} |b_k| \bar{z}^k$$

be in  $VS_H(n, \lambda, \beta)$  and

$$F_n(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + (-1)^{n-1} \sum_{k=1}^{\infty} |B_k| \bar{z}^k$$

be in  $VS_H(n, \lambda, \alpha)$ . Then the convolution  $(f_n * F_n)(z)$  is given by (13). We wish to show that the coefficients of  $(f_n * F_n)(z)$  satisfy the required condition given in Theorem 2.2. For  $F_n(z) \in VS_H(n, \lambda, \alpha)$ , we note that  $|A_k| \leq 1$  and  $|B_k| \leq 1$ . Now, for the convolution function  $(f_n * F_n)(z)$ , we have

$$\begin{aligned} & \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)} \right) |a_k A_k| \\ & \quad + \sum_{k=1}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)} \right) |b_k B_k| \\ & \leq \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)} \right) |a_k| \\ & \quad + \sum_{k=1}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)} \right) |b_k| \\ & \leq 1, \quad \text{since } f_n(z) \in VS_H(n, \lambda, \beta). \end{aligned}$$

Therefore  $(f_n * F_n)(z) \in VS_H(n, \lambda, \beta) \subset VS_H(n, \lambda, \alpha)$ .  $\square$

Next, we show that the class  $VS_H(n, \lambda, \beta)$  is closed under convex combinations of its members.

**Theorem 2.6.** The class  $VS_H(n, \lambda, \beta)$  is closed under convex combination.

*Proof.* For  $i = 1, 2, 3, \dots$  let  $f_{n_i}(z) \in VS_H(n, \lambda, \beta)$ , where  $f_{n_i}(z)$  is given by

$$f_{n_i}(z) = z + \sum_{k=2}^{\infty} |a_{k_i}| z^k + (-1)^{n-1} \sum_{k=1}^{\infty} |b_{k_i}| \bar{z}^k.$$

For  $\sum_{i=1}^{\infty} t_i = 1$ ,  $0 \leq t_i \leq 1$ , the convex combination of  $f_{n_i}$  may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z + \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{k_i}| \right) z^k + (-1)^{n-1} \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{k_i}| \right) \bar{z}^k.$$

Then by Theorem 2.2, we have

$$\begin{aligned} & \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)} \right) \left( \sum_{i=1}^{\infty} t_i |a_{k_i}| \right) \\ & + \sum_{k=1}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)} \right) \left( \sum_{i=1}^{\infty} t_i |b_{k_i}| \right) \\ & = \sum_{i=1}^{\infty} t_i \left( \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)} \right) |a_{k_i}| \right. \\ & \quad \left. + \sum_{k=1}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)} \right) |b_{k_i}| \right) \\ & \leq \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

Therefore  $\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in VS_H(n, \lambda, \beta)$ . □

### 3. A FAMILY OF CLASS PRESERVING INTEGRAL OPERATOR

Let  $f(z) = h(z) + \overline{g(z)}$  be defined by (1) then  $F(z)$  defined by the relation

$$(14) \quad F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t) dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t) dt}, \quad (c > -1).$$

**Theorem 3.1.** Let  $f_n(z) = h(z) + \overline{g_n(z)} \in S_H$  be given by (6) and  $f_n(z) \in VS_H(n, \lambda, \beta)$  then  $F(z)$  be defined by (14) also belong to  $VS_H(n, \lambda, \beta)$ .

*Proof.* From the representation of (14) of  $F(z)$ , it follows that

$$(15) \quad F(z) = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} |a_k| z^k + (-1)^{n-1} \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k| \bar{z}^k.$$

Since  $f_n(z) \in VS_H(n, \lambda, \beta)$ , then by Theorem 2.2 we have

$$\sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)} \right) |a_k| + \sum_{k=1}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)} \right) |b_k| \leq 1.$$

Now

$$\begin{aligned} & \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)} \right) \frac{c+1}{c+k} |a_k| \\ & + \sum_{k=1}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)} \right) \frac{c+1}{c+k} |b_k| \\ & \cdot \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k-\beta+\lambda(1-\beta)}{(\beta-1)(1+\lambda)} \right) |a_k| \\ & + \sum_{k=1}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n \left( \frac{k+\beta+\lambda(1+\beta)}{(\beta-1)(1+\lambda)} \right) |b_k| \\ & \leq 1. \end{aligned}$$

Thus  $F(z) \in VS_H(n, \lambda, \beta)$ . □

## REFERENCES

- [1] O.P. Ahuja, *Planar harmonic univalent and related mappings*, J. Inequal. Pure Appl. Math., **6** (4) (2005), Art. 122, 1-18.
- [2] J. Clunie, T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fen. Series AI Math., **9** (1984), 3-25.
- [3] N.E. Cho, H.M. Srivastava, *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modell., **37**(1-2), (2003), 39-49.
- [4] N.E. Cho, T.H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc., **40**(3)(2003), 399-410.
- [5] K.K. Dixit, Vikas Chandra, *On subclass of univalent functions with positive coefficients*, Aligarh Bull. Math., **27** (2) (2008), 87-93.
- [6] K.K. Dixit, Saurabh Porwal, *A subclass of harmonic univalent functions with positive coefficients*, Tamkang J. Math., **41**(3) (2010), 261-269.
- [7] K.K. Dixit, Saurabh Porwal, Ankit Dixit, *A new subclass of univalent functions with positive coefficients*, Bessel J. Math., **3**(2), (2013), 125-135.
- [8] P. Duren, *Harmonic mappings in the plane*, Camb. Univ. Press, (2004).
- [9] J.M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, *Salagean-type harmonic univalent functions*, Southwest J. Pure Appl. Math., **2** (2002), 77-82.

- 
- [10] A. L. Pathak, S. Porwal, R. Agarwal, R. Misra, *A subclass of harmonic univalent functions with positive coefficients associated with fractional calculus operator*, J. Non-linear Anal. Appl., (2012), Article ID jnaa-00108, 11 Pages.
- [11] Saurabh Porwal, *Study of certain classes related to analytic and harmonic univalent functions*, Ph.D. Thesis, CSJM University, Kanpur, India, 2009.
- [12] Saurabh Porwal, M.K. Aouf, *On a new subclass of harmonic univalent functions defined by fractional calculus operator*, J. Frac. Calc. Appl., **4**(2013), No. 10, 1-12.
- [13] Saurabh Porwal, K.K. Dixit, *New subclasses of harmonic starlike and convex functions*, Kyungpook Math. J., **53**(2013), 467-478.
- [14] Saurabh Porwal, K.K. Dixit, *On a new subclass of Salagean-type harmonic univalent functions*, Indian J. Math., **54**(2) (2012), 199-210.
- [15] Saurabh Porwal, K.K. Dixit, *An application of certain convolution operator involving hypergeometric functions*, J. Raj. Acad. Phy. Sci., **9**(2), (2010), 173-186.
- [16] Saurabh Porwal, K.K. Dixit, A.L. Pathak, R. Agarwal, *A subclass of harmonic univalent functions with positive coefficients defined by generalized Salagean Operator*, J. Raj. Acad. Phy. Sci., **11**(2)(2012), 93-102.
- [17] Saurabh Porwal, K.K. Dixit, Vinod Kumar, Poonam Dixit, *On a subclass of analytic functions defined by convolution*, General Mathematics, **19**(3), (2011), 57-65.
- [18] G.S. Salagean, *Subclasses of univalent functions*, Complex Analysis-Fifth Romanian Finish Seminar, Bucharest, **1** (1983), 362-372.
- [19] B.A. Uralegaddi, C. Somanatha, *Certain classes of univalent functions*, in Current topics in analytic function theory, 371-374, World Sci. Publishing, River Edge, NJ.
- [20] B.A. Uralegaddi, M.D. Ganigi, S.M. Sarangi, *Univalent functions with positive coefficients*, Tamkang J. Math., **25** (3) (1994), 225-230.

**SAURABH PORWAL**

DEPARTMENT OF MATHEMATICS

UIET, C.S.J.M. UNIVERSITY

KANPUR-208024 (U.P.)

INDIA

*E-mail address:* saurabhjcb@rediffmail.com