On \((\sigma, \delta) - (S, 1)\) rings and their extensions

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Abstract. Let \(R\) be a ring, \(\sigma\) an endomorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\). We recall that \(R\) is called an \((S, 1)\)-ring if for \(a, b \in R\), \(ab = 0\) implies \(aRb = 0\). We involve \(\sigma\) and \(\delta\) to generalize this notion and say that \(R\) is a \((\sigma, \delta) - (S, 1)\)-ring if for \(a, b \in R\), \(ab = 0\) implies \(aRb = 0\), \(\sigma(a)Rb = 0\), \(aR\sigma(b) = 0\) and \(\delta(a)Rb = 0\). In case \(\sigma\) is identity, \(R\) is called a \(\delta - (S, 1)\)-ring.

In this paper we study the associated prime ideals of Ore extension \(R[x; \sigma, \delta]\) and we prove the following in this direction:

Let \(R\) be a semiprime right Noetherian ring, which is also an algebra over \(\mathbb{Q}\) (\(\mathbb{Q}\) is the field of rational numbers), \(\sigma\) an automorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\) such that \(R\) is a \((\sigma, \delta) - (S, 1)\)-ring. Then \(P\) is an associated prime ideal of \(R[x; \sigma, \delta]\) (viewed as a right module over itself) if and only if there exists an associated prime ideal \(U\) of \(R\) (viewed as a right module over itself) such that \((P \cap R)[x; \sigma, \delta] = P\) and \(P \cap R = U\).

1. Introduction

All rings are associative with identity \(1 \neq 0\) unless otherwise stated. The ring of integers, the field of rational numbers and the field of real numbers are denoted by \(\mathbb{Z}\), \(\mathbb{Q}\) and \(\mathbb{R}\) respectively, unless otherwise stated. \(\text{Spec}(R)\) denotes the set of prime ideals of \(R\). \(\text{MinSpec}(R)\) denotes the set of minimal prime ideals of \(R\). The Prime radical and the set of nilpotent elements of \(R\) are denoted by \(P(R)\) and \(N(R)\) respectively. For any subset \(J\) of a right \(R\)-module \(M\), annihilator of \(J\) is denoted by \(\text{Ann}(J)\). The set of associated prime ideals of \(R\) (viewed as a right module over itself) is denoted by \(\text{Ass}(R_R)\). Let \(R\) be a right Noetherian ring. For any uniform right \(R\)-module \(J\), the assassinator of \(J\) is denoted by \(\text{Assas}(J)\). Let \(M\) be a right \(R\)-module. Consider the set

\[\{\text{Assas}(J) \mid J \text{ is a uniform right } R\text{-submodule of } M\}\].

We denote this set by \(A(M_R)\).

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Remark 1.1. If $R$ is viewed as a right module over itself, we note that $Ass(R_R) = \mathcal{A}(R_R)$ (5Y of Goodearl and Warfield [3]).

Endomorphisms and derivations.

Let $R$ be a ring, $\sigma$ be an endomorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$. Then $\delta : R \to R$ is an additive map such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$.

Example 1.1. (1) Let $\sigma$ be an automorphism of a ring $R$ and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix},$$

for all $r \in R$.

Then $\delta$ is a $\sigma$-derivation of $R$ if and only if $\phi$ is a homomorphism.

(2) For any endomorphism $\tau$ of a ring $R$ and for any $a \in R$, $\varrho : R \to R$ defined as $\varrho(r) = ra - a\tau(r)$ is a $\tau$-derivation of $R$.

In case $\sigma$ is the identity map, $\delta$ is called just a derivation of $R$. For example let $F$ be a field and $R = F[X]$. Then the usual differential operator $\frac{d}{dx}$ is a derivation of $R$.

Ore extensions.

Ore extension (skew polynomial ring) over $R$ in an indeterminate $x$ is: $R[x; \sigma, \delta] = \{ f(x) = \sum_{i=0}^{n} x^i a_i | a_i \in R \}$ with addition as usual and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x; \sigma, \delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$ as followed in McConnell and Robson [6]. This definition of non-commutative polynomial rings was first introduced by Ore in 1933, who combined earlier ideas of Hilbert (in the case $\delta = 0$) and Schlessinger (in the case $\sigma = 1$). We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. An ideal $I$ of a ring $R$ is called $\sigma$-stable if $\sigma(I) \subseteq I$ and is called $\delta$-invariant if $\delta(I) \subseteq I$. If an ideal $I$ of $R$ is $\sigma$-stable and $\delta$-invariant, then $I[x; \sigma, \delta]$ is an ideal of $O(R)$ and as usual we denote it by $O(I)$.

Theorem 1.1. (Hilbert Basis Theorem, namely Theorem 2.6 of Goodearl and Warfield [3]). Let $R$ be a right/left Noetherian ring. Let $\sigma$ and $\delta$ be as above. Then the Ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian.

2. Preliminaries

Completely prime ideals.

Definition 2.1. (McCoy [7]) An ideal $P$ of a ring $R$ is said to be completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. 

Let \( R = \left( \mathbb{Z} \; \mathbb{Z} \atop 0 \; \mathbb{Z} \right) \). Then \( P = \left( \mathbb{Z} \; \mathbb{Z} \atop 0 \; 0 \right) \) is completely prime ideal of \( R \).

**Note:** In commutative case completely prime ideal and prime have the same meaning. In general (non-commutative) situation every completely prime ideal of a ring \( R \) is a prime ideal, but converse is not be true.

Let \( R = \left( \mathbb{Z} \; \mathbb{Z} \atop \mathbb{Z} \; \mathbb{Z} \right) = M_2(\mathbb{Z}) \). If \( p \) is a prime number, then the ideal \( P = M_2(p\mathbb{Z}) \) is a prime ideal of \( R \), but is not completely prime, since for \( a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), we have \( ab \in P \), even though \( a \notin P \) and \( b \notin P \).

\( \sigma(\ast) \)-rings.

**Definition 2.2.** (Kwak [4]). Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \). Then \( R \) is said to be a \( \sigma(\ast) \)-ring if \( a \sigma(a) \in P(R) \) implies \( a \in P(R) \) for \( a \in R \).

**Example 2.1.** Let \( R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \; ; \; a, \; b, \; c \in F, \; a \text{ field} \right\} \).

Now \( P(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \; ; \; a, \; c \in F \right\} \).

Define \( \sigma : R \to R \) a map by \( \sigma \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \left( \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \right) \). Then \( \sigma \) is an endomorphism of \( R \), and it can be easily seen that \( R \) is a \( \sigma(\ast) \)-ring.

**Remark 2.1.** It can be seen that a \( \sigma(\ast) \)-ring \( R \) is 2-primal, for let \( a \in R \) be such that \( a^2 \in P(R) \). Then

\[ a \sigma(a) \sigma(a) = a \sigma(a)^2(a) \in \sigma(P(R)) = P(R). \]

Therefore, \( a \sigma(a) \in P(R) \) and hence \( a \in P(R) \). So \( P(R) \) is completely semiprime and hence \( R \) is 2-primal.

**Weak \( \sigma \)-rigid rings.**

Ouyang in [8] introduced weak \( \sigma \)-rigid rings, where \( \sigma \) is an endomorphism of ring \( R \). These rings are related to 2-primal rings.

**Definition 2.3.** (Ouyang [8]). Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \) such that \( a \sigma(a) \in N(R) \) if and only if \( a \in N(R) \) for \( a \in R \). Then \( R \) is called a weak \( \sigma \)-rigid ring.

**Example 2.2.** Assume that \( W_1[F] \) is the first Weyl algebra over a field \( F \) of characteristic zero. Then \( W_1[F] = F[\mu, \lambda] \), the polynomial ring with
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indeterminates \(\mu\) and \(\lambda\) with \(\lambda \mu = \mu \lambda + 1\).

Now let \(R\) be the ring

\[
\begin{pmatrix}
W_1[F] & W_1[F] \\
0 & 0
\end{pmatrix}.
\]

Now the prime radical \(P(R)\) of \(R\) is

\[
\begin{pmatrix}
0 & W_1[F] \\
0 & 0
\end{pmatrix}.
\]

Define an endomorphism \(\sigma : R \to R\) by

\[
\sigma \left( \begin{pmatrix}
\mu & \lambda \\
0 & 0
\end{pmatrix} \right) = \begin{pmatrix}
\mu & 0 \\
0 & 0
\end{pmatrix}.
\]

Then \(R\) is a weak \(\sigma\)-rigid ring.

3. \((\sigma, \delta) - (S, 1)\) rings

**Definition 3.1.** (Kim and Lee [5]). A ring \(R\) is called an \((S, 1)\)-ring if for \(a, b \in R\), \(ab = 0\) implies \(aRb = 0\).

This notion was actually introduced by Shin (i.e. a ring satisfying \(SI\) property, Lemma 1.2 of [9]). In this article we generalize the notion of \((S, 1)\)-rings by involving a derivation \(\delta\) of \(R\).

**Definition 3.2.** Let \(R\) be a ring and \(\delta\) a derivation of \(R\). Then \(R\) is called a \(\delta - (S, 1)\) ring if for \(a, b \in R\), \(ab = 0\) implies that \(aRb = 0\) and \(\delta(a)Rb = 0\).

**Example 3.1.** Let \(R = S \times S\), \(S\) a ring and \(A = (u, v) \in R\). Define \(\delta : R \to R\) by \(\delta_A(a,b) = (au - ua, bv - vb)\). Then \(\delta_A\) is a derivation of \(R\).

Now the only elements \(\alpha, \beta \in R\) such that \(\alpha \beta = 0\) are of the form \(\alpha = (a, 0)\), and \(\beta = (0, b)\) and for all \(\gamma = (r, s) \in R\),

\[
\alpha \gamma \beta = 0 \quad \text{and} \quad \delta_A(\alpha) \gamma \beta = \delta((a, 0))(r, s)(0, b) = (au - ua, 0)(r, s)(0, b) = ((au - ua)r, 0)(0, b) = (0, 0)
\]

So, \(R\) is a \(\delta_A - (S, 1)\) ring.

We note that a \(\delta - (S, 1)\) ring is an \((S, 1)\)-ring.

**Definition 3.3.** Let \(R\) be a ring (not necessarily with 1), \(\sigma\) an endomorphism of \(R\) and \(\delta\) a \(\sigma\) derivation of \(R\). Then \(R\) is called a \((\sigma, \delta) - (S, 1)\) ring if for \(a, b \in R\), \(ab = 0\) implies that \(aRb = 0\), \(\sigma(a)Rb = 0\), \(aR\sigma(b) = 0\) and \(\delta(a)Rb = 0\). In case \(\delta\) is the zero map, \((\sigma, \delta) - (S, 1)\) is called a \(\sigma - (S, 1)\).
Example 3.2. Let $S$ be a ring ((not necessarily with 1) and $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\}$; $a, b \in S$).

Define $\sigma : R \to R$ a map by $\sigma \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then $\sigma$ is an endomorphism of $R$.

Define $\delta : R \to R$ a map by $\delta \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$. Then $\delta$ is a $\sigma$-derivation of $R$.

Now the only matrices $A \in R$ and $B \in R$ satisfying $AB = 0$ are of the type $A = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}$; $u, v \in \mathbb{H}$.

Now for all $J = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in R$,

$AJB = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

and

(1) $\sigma(A)JB = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

(2) $AJ\sigma(B) = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

(3) $\delta(A)JB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Therefore, $R$ is a $(\sigma, \delta) - (S, 1)$ ring.

By definition, we see that a $(\sigma, \delta) - (S, 1)$ ring is an $(S, 1)$ ring, but the converse is not true.

Example 3.3. Let $S$ be a domain and $R = S \times S$. Let $\sigma : R \to R$ be defined by $\sigma(a, b) = (b, a)$. Then $\sigma$ is an endomorphism of $R$. Let $A = (u, v) \in R$, define $\delta_A : R \to R$ a map by

$\delta_A((a, b)) = (au - ub, bv - va)$

Then $\delta_A$ is a $\sigma$-derivation of $R$.

Now the only elements $p, q \in R$ such that $pq = 0$ are of the form $p = (a, 0)$ and $q = (0, b)$, for all $a, b \in S$. Now for all $t = (r, s) \in R$, $ptq = (0, 0)$. So $R$ is an $(S, 1)$-ring, but for nonzero $a, b, u, v, r, s \in S$,

$\sigma(p)tq = \sigma((a, 0))(r, s)(0, b) = (0, a)(r, s)(0, b) = (0, asb) \neq 0$,

$pt\sigma(q) = (a, 0)(r, s)\sigma((0, b)) = (a, 0)(r, s)(b, 0) = (arb, 0) \neq 0$, and

$\delta_A(p)tq = \delta_A((a, 0))(r, s)(0, b) = (au, -va)(r, s)(0, b) = (0, -vasb) \neq 0$. 
Thus, $R$ is not a $(\sigma, \delta) - (S, 1)$ ring.

4. Completely prime ideals of $(\sigma, \delta) - (S, 1)$ rings

Proposition 4.1. Let $R$ be a ring, $\sigma$ an automorphism of $R$ such that $R$ is a $\sigma - (S, 1)$ ring. Then $R$ is 2-primal.

Proof. $R$ is a $\sigma - (S, 1)$ ring. Therefore by Theorem 1.5 of [9] $R$ is 2-primal, which implies that $P(R)$ is completely semiprime. We give a sketch of proof.

Let $a \in N(R)$, say $a^n = 0$. If $a \notin P$ for some prime ideal $P$, then $ax_1a \notin P$ for some element $x_1 \in R$. Continuing the process we can find elements $x_i \in R$ such that $P$ does not contain $b = ax_1a...x_{n-1}a$. But, $R$ is an $(S, 1)$-ring, so $a^n = 0$ implies $b = 0$, hence $b \in P$, a contradiction. □

Proposition 4.2. Let $R$ be a ring, $\sigma$ an automorphism of $R$ such that $R$ is a $\sigma - (S, 1)$ ring. Then $R$ is a $\sigma(\ast)$-ring.

Proof. $R$ is 2-primal and $P(R)$ is completely semiprime by Proposition 4.1.

We will show that $R$ is a weak $\sigma$-rigid ring. Let $a \in R$ be such that $a\sigma(a) \in N(R)$. Now $a\sigma(a)\sigma^{-1}(a\sigma(a)) \in N(R)$ implies that $a^2 \in N(R)$, and so $a \in N(R)$. Therefore, $R$ is a weak $\sigma$-rigid ring, and is also a $\sigma(\ast)$-ring. □

Remark 4.1. Converse of Proposition 4.1 and Proposition 4.2 is not true. For example, the ring in Example 2.1 is a $\sigma(\ast)$-ring and, therefore, is also a 2-primal ring, but it is not a $\sigma - (S, 1)$ ring (even not an $(S, 1)$ ring).

Proposition 4.3. Let $R$ be a right Noetherian which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $R$ is a $(\sigma, \delta) - (S, 1)$ ring. Then $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in \text{MinSpec}(R)$.

Proof. $R$ is 2-primal by Proposition 4.1 and is a $\sigma(\ast)$-ring by Proposition 4.2.

Now the result follows by Proposition 2.1 of [2]. □

Proposition 4.4. Let $R$ be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $R$ is a $(\sigma, \delta) - (S, 1)$ ring. Then $U \in \text{MinSpec}(R)$ implies that $U$ is a completely prime ideal of $R$. 


Proof. Suppose that \( U \) is not completely prime. Then there exist \( a, b \in R \setminus U \) with \( ab \in U \). Consider \( U_i \) as in 4.3. Let \( c \) be any element of \( b(U_2 \cap U_3 \cap \ldots \cap U_n)a \). Then \( c^2 \in \bigcap_{i=1}^n U_i = P(R) \). So \( c \in P(R) \) and, thus \( b(U_2 \cap U_3 \cap \ldots \cap U_n)a \subseteq U \). Therefore, \( bR(U_2 \cap U_3 \cap \ldots \cap U_n)Ra \subseteq U \) and, as \( U \) is prime, \( a \in U \), \( U_i \subseteq U \) for some \( i \neq 1 \) or \( b \in U \). None of these can occur, so \( U \) is completely prime. \( \square \)

**Proposition 4.5.** Let \( R \) be a right Noetherian ring which is also an algebra over \( \mathbb{Q} \). Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \) such that \( R \) is a \( (\sigma, \delta) - (S, 1) \) ring. Then \( U \in \text{Min.Spec}(R) \) implies that \( O(U) \) is a completely prime ideal of \( O(R) \).

**Proof.** Let \( U \in \text{Min.Spec}(R) \). Now Proposition 4.3 implies that \( \sigma(U) = U \) and \( \delta(U) \subseteq U \). Also Now Proposition 4.4 implies that \( U \) is a completely prime ideal of \( R \). Now the result follows by Theorem 2.4 of [1]. \( \square \)

**Theorem 4.1.** Let \( R \) be a semiprime right Noetherian ring. Let \( \sigma \) be an automorphism of \( R \) such that \( R \) is a \( (\sigma, \delta) - (S, 1) \) ring. Then \( P \in \text{Ass}(O(R)_{O(R)}) \) if and only if there exists \( U \in \text{Ass}(R_R) \) such that \( O(P \cap R) = P \) and \( P \cap R = U \).

**Proof.** \( R \) being right Noetherian implies that \( \text{Ass}(R_R) = \mathbb{A}(R_R) \) (Remark 1.1). Now \( R \) a \( (\sigma, \delta) - (S, 1) \) ring implies that \( \sigma(U) = U \) and \( \delta(U) \subseteq U \) for all \( U \in \text{Min.Spec}(R) \) by Proposition 4.3.

\( O(R) \) is right Noetherian by Hilbert Basis Theorem. Let \( J \in \text{Ass}(O(R)_{O(R)}) \). Now by Remark 1.1 \( \text{Ass}(O(R)_{O(R)}) = \mathbb{A}(O(R)_{O(R)}) \). Let \( P = \text{Ann}(I) = \text{Assas}(I) \) for some ideal \( I \) of \( O(R) \) such that \( I \) is uniform as a right \( O(R) \)-module. Choose \( f \in I \) to be nonzero of minimal degree (with leading coefficient \( a_n \)). Let \( U = \text{Ann}(a_nR) = \text{Assas}(a_nR) \). Now \( R \) is right Noetherian implies that \( \text{Ass}(R_R) = \mathbb{A}(R_R) \) and since \( R \) is semiprime, \( U \in \text{Min.Spec}(R) \) by Proposition (2.2.14) of McConnell and Robson [6]. Now \( \sigma(U) = U \), and \( \delta(U) \subseteq U \) by Proposition 4.3. So \( O(U) \) is an ideal of \( O(R) \). Now \( fU = 0 \). Therefore \( fO(R)U \subseteq fUO(R) = 0 \). So \( U \subseteq P \cap R \). But it is clear that \( P \cap R \subseteq U \). Thus \( P \cap R = U \).

Conversely let \( U = \text{Ann}(cR) = \text{Assas}(cR) \), \( c \in R \). Now \( R \) is right Noetherian implies that \( \text{Ass}(R_R) = \mathbb{A}(R_R) \). Now \( \sigma(U) = U \), \( \delta(U) \subseteq U \) and it can be easily seen that \( O(U) = \text{Ann}(hO(R)) \) for all \( h \in O(R) \). Therefore \( O(U) = \text{Ann}(cO(R)) = \text{Assas}(cO(R)) \). \( \square \)
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