A generalization of modules with the property \((P^*)\)

BURCU NIŞANCI TÜRKMEN

Abstract. I.A. Khazzi and P.F. Smith called a module \(M\) have the property \((P^*)\) if every submodule \(N\) of \(M\) there exists a direct summand \(K\) of \(M\) such that \(K \leq N\) and \(N^K \subseteq \text{Rad}(M/R)\). Motivated by this, it is natural to introduce another notion that we called modules that have the properties \((GP^*)\) and \((N - GP^*)\) as proper generalizations of modules that have the property \((P^*)\). In this paper we obtain various properties of modules that have properties \((GP^*)\) and \((N - GP^*)\). We show that the class of modules for which every direct summand is a fully invariant submodule that have the property \((GP^*)\) is closed under finite direct sums. We completely determine the structure of these modules over generalized f-semiperfect rings.

1. Introduction

Throughout this paper, all rings are associative with identity element and all modules are unital right \(R\)-modules. Let \(R\) be a ring and let \(M\) be an \(R\)-module. The notation \(N \leq M\) means that \(N\) is a submodule of \(M\). A module \(M\) is called extending if every submodule of \(M\) is essential in a direct summand of \(M\) [4]. Here a submodule \(L \leq M\) is said to be essential in \(M\), denoted as \(L \subseteq M\), if \(L \cap N \neq 0\) for every nonzero submodule \(N \leq M\). Dually, a submodule \(S\) of \(M\) is called small \((in M)\)), denoted as \(S << M\), if \(M = S + L\) for every proper submodule \(L\) of \(M\). By \(\text{Rad}(M)\), we denote the intersection of all maximal submodules of \(M\). An \(R\)-module \(M\) is called supplemented if every submodule \(N\) of \(M\) has a supplement, that is a submodule \(K\) minimal with respect to \(M = N + K\). Equivalently, \(M = N + K\) and \(N \cap K << K\) [11]. \(M\) is called \((f-)\) supplemented if every (finitely generated) submodule of \(M\) has a supplement in \(M\) (see [11]). On the other hand, \(M\) is called amply supplemented if, for any submodules \(N\) and \(K\) of \(M\) with \(M = N + K\), \(K\) contains a supplement of \(N\) in \(M\). Accordingly a module \(M\) is called amply \(f\)-supplemented if every finitely generated submodule of \(M\) satisfies same condition. It is clear that (amply) \(f\)-supplemented modules are a proper generalization of (amply) supplemented modules.

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A module $M$ is called lifting if for every submodule $N$ of $M$ there exists a direct summand $K$ of $M$ such that $K \leq N$ and $\frac{N}{K} \ll \frac{M}{K}$ (i.e. $K$ is a coessential submodule of $N$ in $M$) as a dual notion of extending modules. Mohamed and Müller has generalized the concept of lifting modules to $\oplus$-supplemented modules. $M$ is called $\oplus$-supplemented if every submodule of $M$ has a supplement that is a direct summand of $M$ [6].

Let $M$ be an $R$-module and let $N$ and $K$ be any submodules of $M$. If $M = N + K$ and $N \cap K \subseteq \text{Rad}(K)$, then $K$ is called a Rad-supplement of $N$ in $M$ [12](according to [10], generalized supplement). It is clear that every supplement is Rad-supplement. $M$ is called Rad-supplemented (according to [10], generalized supplemented) if every submodule of $M$ has a Rad-supplement in $M$, and $M$ is called amply Rad-supplemented if, for any submodules $N$ and $K$ of $M$ with $M = N + K$, $K$ contains a Rad-supplement of $N$ in $M$. An $R$-module $M$ is called $f$-Rad-supplemented if every finitely generated submodule of $M$ has a Rad-supplement in $M$, and a module $M$ is called amply $f$-Rad-supplemented if every finitely generated submodule of $M$ has ample $f$-Rad-supplements in $M$ (see [7]). A module $M$ is called Rad-$\oplus$-supplemented if every submodule has a Rad-supplement that is a direct summand of $M$ [3] and [5].

Recall from Al-Khazzi and Smith [1] that a module $M$ is said to have the property $(P^*)$ if for every submodule $N$ of $M$ there exists a direct summand $K$ of $M$ such that $K \leq N$ and $\frac{N}{K} \subseteq \text{Rad}(\frac{M}{K})$. The authors have obtained in the same paper the various properties of modules with the property $(P^*)$. Radical modules have the property $(P^*)$. It is clear that every lifting module has the property $(P^*)$ and every module with the property $(P^*)$ is Rad-$\oplus$-supplemented.

Let $f : P \longrightarrow M$ be an epimorphism. If $\text{Ker}(f) \ll P$, then $f$ is called cover, and if $P$ is a projective module, then a cover $f$ is called a projective cover [11]. Xue [12] calls $f$ a generalized cover if $\text{Ker}(f) \leq \text{Rad}(P)$, and calls a generalized cover $f$ a generalized projective cover if $P$ is a projective module. In the spirit of [12], a module $M$ is said to be (generalized) semiperfect if every factor module of $M$ has a (generalized) projective cover. A module $M$ is said to be $f$-semiperfect if, for every finitely generated submodule $U \leq M$, the factor module $\frac{M}{U}$ has a projective cover in $M$ [11]. Let $M$ be an $R$-module. $M$ is called generalized $f$-semiperfect module if, for every finitely generated submodule $U \leq M$, the factor module $\frac{M}{U}$ has a generalized projective cover in $M$ [8].

In this study, we obtain some elementary facts about the properties $(GP^*)$ and $(N - GP^*)$ which are a proper generalizations of the property $(P^*)$. Especially, we give a relation for $G^*-$supplemented modules. We prove that every direct summand of a module that have the property $(GP^*)$ has the property $(GP^*)$. We show that a module $M$ has the property $(N - GP^*)$ if and only if, for all direct summands $M'$ and a coclosed submodule $N'$ of $N$,
$M'$ has the property $(N' - GP^*)$ for right $R$-modules $M$ and $N$. We obtain
that Let $M = \bigoplus_{i=1}^{n} M_i$ be a module and $M_i$ is a fully invariant submodule
of $M$ for all $i \in \{1, 2, \ldots, n\}$. Then $M$ has the property $(GP^*)$ if and only if
$M_i$ has the property $(GP^*)$ for all $i \in \{1, 2, \ldots, n\}$. We illustrate a module
with the property $(GP^*)$ which doesn’t have the property $(P^*)$. We give a
characterization of generalized f-semiperfect rings via the property $(GP^*)$.

2. Modules with the Properties of $(GP^*)$ and $(N - GP^*)$

**Definition 2.1.** A module $M$ has the property $(GP^*)$ if, for every $\gamma \in \text{End}_R(M)$ there exists a direct summand $N$ of $M$ such that $N \subseteq \text{Im}(\gamma)$ and $\frac{\text{Im}(\gamma)}{N} \subseteq \text{Rad}(\frac{M}{N})$.

**Proposition 2.1.** The following conditions are equivalent for a module $M$.

1. $M$ has the property $(GP^*)$.
2. For every $\gamma \in \text{End}_R(M)$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq \text{Im}(\gamma)$ and $M_2 \cap \text{Im}(\gamma) \subseteq \text{Rad}(M_2)$.
3. For every $\gamma \in \text{End}_R(M)$, $\text{Im}(\gamma)$ can be represented as $\text{Im}(\gamma) = N \oplus N'$, where $N$ is a direct summand of $M$ and $N' \subseteq \text{Rad}(M)$.

**Proof.** (1) $\Rightarrow$ (2) By the hypothesis, there exist direct summands $M_1$, $M_2$ of $M$ such that $M_1 \subseteq \text{Im}(\gamma)$, $M = M_1 \oplus M_2$ and $\text{Im}(\gamma) = \text{Rad}(\frac{M_1}{M_2}) \cdot \text{Rad}(M_2)$ since $M_2$ is a Rad-supplement of $M_1$ in $M$, $\text{Rad}(\frac{M}{M_1} \oplus M_2) = \frac{\text{Rad}(M) + M_1}{M_1}$ (See [13, Lemma 1.1]). Then $\frac{\text{Im}(\gamma)}{M_1} \subseteq \frac{\text{Rad}(M) + M_1}{M_1}$. So we have $\text{Im}(\gamma) \subseteq \text{Rad}(M_2) + M_1$. By the modular law, $M_2 \cap \text{Im}(\gamma) \subseteq \text{Rad}(M_2)$.

(2) $\Rightarrow$ (3) For every $\gamma \in \text{End}_R(M)$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq \text{Im}(\gamma)$ and $M_2 \cap \text{Im}(\gamma) \subseteq \text{Rad}(M_2)$. So $\text{Im}(\gamma) = M_1 \oplus (\text{Im}(\gamma) \cap M_2)$ by the modular law. Say $N = M_1$ and $N' = \text{Im}(\gamma) \cap M_2$. Therefore $\text{Im}(\gamma) = N \oplus N'$, where $N$ is a direct summand of $M$ and $N' \subseteq \text{Rad}(M)$.

(3) $\Rightarrow$ (1) By the hypothesis, for every $\gamma \in \text{End}_R(M)$, $\text{Im}(\gamma) = N \oplus N'$ where $N$ is a direct summand of $M$ and $N' \subseteq \text{Rad}(M)$. Thus there exists a direct summand $N$ of $M$ such that $N \subseteq \text{Im}(\gamma)$. We have $\frac{\text{Im}(\gamma)}{N} = \frac{N \oplus N'}{N} \subseteq \frac{N + \text{Rad}(M)}{N} \subseteq \text{Rad}(\frac{M}{N})$. \hfill $\square$

**Definition 2.2.** A module $M$ has the property $(N - GP^*)$ if, for every homomorphism $\gamma : M \rightarrow N$, there exists a direct summand $L$ of $N$ such that $L \subseteq \text{Im}(\gamma)$ and $\frac{\text{Im}(\gamma)}{L} \subseteq \text{Rad}(\frac{N}{L})$.

It is clear that a right module $M$ has the property $(GP^*)$ if and only if $M$ has the property $(M - GP^*)$.

Recall from [4, 3.6] that a submodule $N$ of $M$ is called coclosed in $M$ if, $N$ has no proper submodule $K$ for which $K \subset N$ is cosmall in $M$, that is, $\frac{N}{K} \ll \frac{M}{K}$. Obviously any direct summand $N$ of $M$ is coclosed in $M$. 

**Theorem 2.1.** Let $M$ and $N$ be right $R$-modules. Then $M$ has the property $(N - GP^*)$ if and only if, for all direct summands $M'$ and a coclosed submodule $N'$ of $N$, $M'$ has the property $(N' - GP^*)$.

**Proof.** ($\implies$) Let $M' = eM$ for some $e^2 = e \in \text{End}_R(M)$ and let $N'$ be a coclosed submodule of $N$. Assume that $\alpha \in \text{Hom}(M', N')$. Since $\alpha(eM) = \alpha(M') \subseteq N' \subseteq N$ and $M$ has the property $(N - GP^*)$, there exists a decomposition $N = N_1 \oplus N_2$ such that $N_1 \subseteq \text{Im}(\alpha(e))$ and $N_2 \cap \text{Im}(\alpha(e)) \subseteq \text{Rad}(M_2) \subseteq \text{Rad}(N)$. Then we have $N' = N_1 \oplus (N_2 \cap N')$ by the modular law. Since $N'$ is a coclosed submodule of $N$, then $\text{Rad}(N') = \text{Rad}(N) \cap N'$ by [4, 3.7(3)]. So $N_2 \cap N' \cap \text{Im}(\alpha) \subseteq \text{Rad}(N')$. By using [4, 3.7(3)] once again, we get $N_2 \cap N' \cap \text{Im}(\alpha) \subseteq \text{Rad}(N_2 \cap N')$. Therefore $M'$ has the property $(N' - GP^*)$.

($\impliedby$) Clear. 

**Corollary 2.1.** The following conditions are equivalent for a module $M$.

(1) $M$ has the property $(GP^*)$.

(2) For any coclosed submodule $N$ of $M$, every direct summand $L$ of $M$ has the property $(N - GP^*)$.

**Corollary 2.2.** Every direct summand of a module that have the property $(GP^*)$ has the property $(GP^*)$.

**Proposition 2.2.** Let $M$ be an indecomposable module. Assume that, for $\delta \in \text{End}_R(M)$, $\text{Im}(\delta) \subseteq \text{Rad}(M)$ implies $\delta = 0$. Then, $M$ has the property $(GP^*)$ if and only if every nonzero endomorphism $\delta \in \text{End}_R(M)$ is an epimorphism.

**Proof.** Assume that $0 \neq \delta \in \text{End}_R(M)$. Since $M$ has the property $(GP^*)$, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq \text{Im}(\delta)$ and $M_2 \cap \text{Im}(\delta) \subseteq \text{Rad}(M_2)$. Since $M$ is indecomposable, $M_1 = 0$ or $M_1 = M$. If $M_1 = 0$, then $\text{Im}(\delta) \subseteq \text{Rad}(M)$. By the hypothesis $\delta = 0$; a contradiction. Thus, $M_1 = M$ and hence, $\delta$ is epimorphism. The converse is clear. 

Recall from [4, 4.27] that a module $M$ is said to be Hopfian if every surjective endomorphism of $M$ is an isomorphism.

**Proposition 2.3.** Let $M$ be a noetherian module that has the property $(GP^*)$. If every endomorphism $\gamma$ of $M$, $\text{Im}(\gamma) \subseteq \text{Rad}(M)$ implies that $\gamma = 0$. Then there exists a decomposition $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where $M_i$ is an indecomposable noetherian modules that has the property $(GP^*)$ for which $\text{End}_R(M_i)$ is a division ring.

**Proof.** Since $M$ is noetherian, it has a finite decomposition noetherian direct summands. By Corollary 2.2, every direct summand has the property $(GP^*)$. By Proposition 2.2, in view of the fact that every noetherian module is Hopfian, each indecomposable direct summand has a division ring. 

\[\square\]
**Definition 2.3.** A module $M$ is called $G^* -$supplemented if, for every $\gamma \in \text{End}_R(M)$, $\text{Im}(\gamma)$ has a Rad-supplement in $M$, and a module $M$ is called amply $G^* -$supplemented if, for every $\gamma \in \text{End}_R(M)$, $\text{Im}(\gamma)$ has ample Rad-supplements in $M$.

It is clear that every module that has the property $(GP^*)$ is $G^* -$supplemented by the Definition 2.3.

**Proposition 2.4.** Let $M$ be an amply $G^* -$supplemented $R$-module. Then every direct summand of $M$ is amply $G^* -$supplemented.

**Proof.** Let $N$ be a direct summand of $M$. Then $M = N \oplus N'$ for some $N' \subseteq M$. Suppose that $f \in \text{End}_R(N)$ and $N = \text{Im}(f) + K$. Thus, $M = \text{Im}(f) + K + N'$. Note that $\text{Im}(f) = \text{Im}(\iota f \pi)$, where $\iota$ is the injection map from $N$ to $M$ and $\pi$ is the projection map from $M$ onto $N$. Since $M$ is amply $G^* -$supplemented, there exists a Rad-supplement $L$ of $N' + K$ with $L \subseteq \text{Im}(f)$. We get $K \cap L \subseteq (N' + K) \cap L \subseteq \text{Rad}(L)$ and $M = L + N' + K$. Thus $N = K + L$ by the modular law. So $K + L = N$ and $K \cap L \subseteq \text{Rad}(L)$. Therefore $N$ is amply $G^* -$supplemented. 

**Proposition 2.5.** Let $M$ be an amply $G^* -$supplemented distributive module and let $N$ be a direct summand of $M$ for every Rad-supplement submodule $N$ of $M$. Then $M$ is a $G^* -$supplemented module.

**Proof.** Let $f \in \text{End}_R(M)$, let $K$ be a Rad-supplement of $\text{Im}(f)$ in $M$, and let $N$ a Rad-supplement of $K$ in $M$ with $N \subseteq \text{Im}(f)$. By the hypothesis, $M = N \oplus N'$ for some $N' \subseteq M$. $\text{Im}(f) = \text{Im}(f) \cap (N + K) = N + (\text{Im}(f) \cap K)$. Since $\text{Im}(f) \cap K \subseteq \text{Rad}(K)$, then we have $\text{Im}(f) \cap K \cap N' \subseteq \text{Rad}(K)$. As $M$ is distributive, $\text{Im}(f) + K \cap N' = N + K = M$ and $K = K \cap (N \oplus N') = (K \cap N) \oplus (K \cap N')$. So $K \cap N'$ is a direct summand of $K$. Since $\text{Im}(f) \cap K \cap N' \subseteq K \cap N'$, $\text{Im}(f) \cap K \cap N' \subseteq \text{Rad}(K \cap N')$. Therefore $M$ is $G^* -$supplemented.

**Definition 2.4.** A module $M$ is called $N - G^* -$supplemented if, for every homomorphism $\phi : M \rightarrow N$, there exists $L \subseteq N$ such that $\text{Im}(\phi) + L = N$ and $\text{Im}(\phi) \cap L \subseteq \text{Rad}(L)$. It is clear that the right module $M$ is $G^* -$supplemented if and only if $M$ is $M - G^* -$supplemented.

Recall from [11] that a submodule $U$ of an $R$-module $M$ is called fully invariant if $f(U)$ is contained in $U$ for every $R$-endomorphism $f$ of $M$. A module $M$ is called duo, if for every submodule of $M$ is fully invariant [9].

**Theorem 2.2.** Let $M_1, M_2$ and $N$ be modules. If $N$ is $M_i - G^* -$supplemented for $i = 1, 2$, then $N$ is $M_1 \oplus M_2 - G^* -$supplemented. The converse is true if $M_1 \oplus M_2$ is a duo module.

**Proof.** Suppose that $N$ is $M_i - G^* -$supplemented for $i = 1, 2$. We prove that $N$ is $M_1 \oplus M_2 - G^* -$supplemented. Let $\phi = (\pi_1 \phi, \pi_2 \phi)$ be any homomorphism from $N$ to $M_1 \oplus M_2$, where $\pi_i$ is the projection map from $M_1 \oplus M_2$.
into $M_i$ for $i = 1, 2$. Since $N$ is $M_i - G^*$-supplemented, there exists a submodule $K_i$ of $M_i$ such that $\pi_i \phi N + K_i = M_i$ and $\pi_i \phi N \cap K_i \subseteq \text{Rad}(K_i)$ for $i = 1, 2$. Let $K = K_1 \oplus K_2$. Then $M_1 \oplus M_2 = \pi_1 \phi N + \pi_2 \phi N + K_1 + K_2 = \phi N + K$. Since $\phi N \cap (K_1 + K_2) \subseteq (\phi N + K_1) \cap K_2 + (\phi N + K_2) \cap K_1$, we get $\phi N \cap (K_1 + K_2) \subseteq (\phi N + M_1) \cap K_2 + (\phi N + M_2) \cap K_1$. Since $\phi N + M_1 = \pi_2 \phi N \oplus M_1$ and $\phi N + M_2 = \pi_1 \phi N \oplus M_2$, we conclude that $\phi N \cap K \subseteq (\pi_2 \phi N \cap K_2) + (\pi_1 \phi N + K_1)$. Since $\pi_i \phi N \cap K_i \subseteq \text{Rad}(K_i)$ for $i = 1, 2$, we get $\phi N \cap K \subseteq \text{Rad}(K)$. Hence, $N$ is $M_1 \oplus M_2 - G^*$-supplemented.

Conversely, let $N$ be $M_1 \oplus M_2 - G^*$-supplemented. Let $\phi$ be a homomorphism from $N$ to $M_1$. Then $\text{Im}(\iota \phi) = \text{Im}(\phi)$, where $\iota$ is the canonical inclusion from $M_1$ to $M_1 \oplus M_2$. Since $N$ is $M_1 \oplus M_2 - G^*$-supplemented, there exists $K \subseteq M_1 \oplus M_2$ such that $M_1 \oplus M_2 = \text{Im}(\phi) + K$ and $\text{Im}(\phi) \cap K \subseteq \text{Rad}(K)$. Thus, $M_1 = \text{Im}(\phi) + (K \cap M_1)$ and $\text{Im}(\phi) \cap K \cap M_1 = \text{Im}(\phi) \cap K \subseteq \text{Rad}(K)$. As $M_1 \oplus M_2$ is a duo module and $K = K_1 \oplus K_2 \leq M_1 \oplus M_2$, $K \cap M_1$ is a direct summand of $K$. Hence $\text{Im}(\phi) \cap K \cap M \subseteq \text{Rad}(K \cap M_1)$. Therefore $N$ is an $M_1 - G^*$-supplemented.

**Corollary 2.3.** Suppose that $M = M_1 \oplus M_2$ and $M$ is a $G^*$-supplemented module for $i = 1, 2$. Then $M$ is $G^*$-supplemented and, for every $f \in \text{End}_R(M)$, $\text{Im}(f)$ has a Rad-supplement of the form $K_1 + K_2$ with $K_1 \subseteq M_1$ and $K_2 \subseteq M_2$.

**Proof.** Follows from the proof of Theorem 2.2. 

**Theorem 2.3.** Let $M = \bigoplus_{i=1}^n M_i$ be a module and $M_i$ be a fully invariant submodule of $M$ for all $i \in \{1, 2, \ldots, n\}$. Then $M$ has the property $(GP^*)$ if and only if $M_i$ has the property $(GP^*)$ for all $i \in \{1, 2, \ldots, n\}$.

**Proof.** The necessity follows from Theorem 2.1. Conversely, let $N_i$ be a module that have the property $(GP^*)$ for all $i \in \{1, 2, \ldots, n\}$. Also let $\phi = (\phi_{ij})_{i,j \in \{1, 2, \ldots, n\}} \in \text{End}(M)$ be arbitrary, where $(\phi_{ij}) \in \text{Hom}(M_j, M_i)$. Since $M_i$ is a fully invariant submodule of $M$ for all $i \in \{1, 2, \ldots, n\}$, we get $\text{Im}(\phi) = \bigoplus_{i=1}^n \text{Im}(\phi_{ii})$. As $M_i$ has the property $(GP^*)$, there exists a direct summand $N_i$ of $M_i$ and a submodule $K_i$ of $M_i$ with $N_i \subseteq \text{Im}(\phi_{ii})$, $\text{Im}(\phi_{ii}) = N_i + K_i$ and $K_i \subseteq \text{Rad}(M_i)$. We say $N = \bigoplus_{i=1}^n N_i$. Then $N$ is a direct summand of $M$. Moreover, $\text{Im}(\phi) = \bigoplus_{i=1}^n \text{Im}(\phi_{ii}) = \sum_{i=1}^n N_i + \sum_{i=1}^n K_i$ and $\bigoplus_{i=1}^n K_i \subseteq \text{Rad}(\bigoplus_{i=1}^n M_i) = \text{Rad}(M)$. Therefore $M$ has the property $(GP^*)$. 

**Theorem 2.4.** The following assertions are equivalent for a ring $R$.

(1) $R$ is generalized $f$-semiperfect.
(2) $R_R$ is $f$-Rad-supplemented.
(3) Every cyclic right ideal has a Rad-supplement in $R_R$.
(4) $R_R$ is a $G^*$-supplemented module.
(5) $R_R$ has the property $(GP^*)$.

**Proof.** (1) $\iff$ (2) $\iff$ (3) By [8, Theorem 2.22].
(3) ⇒ (4) is clear because $\text{Im}(\gamma)$ is cyclic for every $\gamma \in \text{End}_R(R_R)$.

(4) ⇒ (3) Assume that $I = aR$ is any cyclic right ideal of $R$. Consider the $R$-homomorphism $\phi : R_R \rightarrow R_R$ defined by $\phi(r) = ar$; where $r \in R$. Then $\text{Im}(\phi) = I$. By (4), $\text{Im}(\phi) = I$ has a Rad-supplement in $R_R$.

(5) ⇒ (4) is clear.

(5) ⇒ (3) Suppose that $R_R$ has the property $(GP^*)$. Let $J = bR$ is any cyclic right ideal of $R$. Consider the $R$-homomorphism $\phi : R_R \rightarrow R_R$ defined by $\phi(r) = br$; where $r \in R$. Then $\text{Im}(\phi) = J$. By (5), there exists submodules $R_1, R_2$ of $R_R$ such that $R_R = R_1 \oplus R_2$, $R_1 \subseteq \text{Im}(\phi) = J$ and $R_2 \cap \text{Im}(\phi) \subseteq \text{Rad}(R_2)$. So $R_R = J + R_2$ and $J \cap R_2 \subseteq \text{Rad}(R_2)$. Thus $R_2$ is a Rad-supplement of $J$ in $R_R$. □

The equivalent condition for the property $(P^*)$ if every submodule $N$ of $M$ there exist submodules $K, K'$ of $M$ such that $K \leq N$, $M = K \oplus K'$ and $N \cap K' \subseteq \text{Rad}(K')$ (See [1]).

**Proposition 2.6.** Let $M$ be a module which has the property $(P^*)$. Then $M$ has the property $(GP^*)$.

**Proof.** Let $\phi : M \rightarrow M$ be any homomorphism. Since $M$ has the property $(P^*)$, there exist submodules $K, K'$ of $M$ such that $K \leq \text{Im}(\phi)$, $M = K \oplus K'$ and $\text{Im}(\phi) \cap K' \subseteq \text{Rad}(K')$. So $M$ has the property $(GP^*)$. □

**Example 2.1.** (See [2]) Let $F$ be any field. Consider the commutative ring $R$ which is the direct product $\prod_{i=0}^{\infty} F_i$, where $F_i = F$. So $R_R$ is a regular ring which is not semisimple. The right $R$-module $R$ is f-Rad-supplemented but not Rad-supplemented. Since $R_R$ is f-Rad-supplemented, $R_R$ has the property $(GP^*)$ by Theorem 2.4. As $R_R$ is not Rad-supplemented, $R_R$ has not the property $(P^*)$.

**References**


**Burcu Nişancı Türkmen**

**Amasya University**

**Faculty of Art and Science**

**Department of Mathematics**

**05100 Amasya**

**Turkey**

E-mail address: burcunisancie@hotmail.com