Mathematica Moravica
Vol. **22**, No. **1** (2018), 43–54 doi: 10.5937/MatMor1801043M

Generalized $C_{\beta}^{\psi}-$ rational contraction and fixed point theorem with application to second order differential equation

Naveen Mani

Abstract. In this article, generalized C_{β}^{ψ}- rational contraction is defined and the existence and uniqueness of fixed points for self map in partially ordered metric spaces are discussed. As an application, we apply our result to find existence and uniqueness of solutions of second order differential equations with boundary conditions.

1. Introduction

From last 15 years, several authors have studied and derived various fixed point results for many contractions in partially ordered sets. Ran and Reurings [1] derived a fixed point result on partially ordered sets in which contractive condition assumed to be hold on comparable elements. After that, author in [9, 10] deduced some results to get fixed point for monotone, non-decreasing operator with partially ordered relation on a set Y without using the continuity of maps. They also discussed few applications of their main findings and gave existence as well as uniqueness theorem ordinary differential equation of first order and first degree with restricted boundary conditions. Number of results after that have been investigated to establish fixed point in partially ordered metric spaces (for more detail see [2, 4, 7, 8, 11, 12, 13, 15, 18, 19, 21, 22]).

In 1975, Jaggi [23] and Das and Gupta [24] derived some fixed point results for rational type contraction. There exist several results in the literature for self and pair of maps satisfying rational expression in different spaces [20, 25].

In 2007, Suzuki [16] introduced the weaker C- contractive condition and proved some fixed point theorems. The existence as well as uniqueness of fixed point of such types of operator have also been extensively studied in [3, 17].

2010 *Mathematics Subject Classification.* Primary: 47H10; Secondary: 54H25.

Key words and phrases. Fixed point, $C_{\beta}^{\psi}-$ rational contraction, partially ordered metric spaces, differential equations.

Full paper. Received 8 January 2018, revised 25 February 2018, accepted 11 March 2018, available online 25 June 2018.

Definition 1.1. [16] Let (Y, d) be a metric space. Then a map f on Y is said to satisfies the $C-$ condition if, for all $u, v \in Y$,

$$\frac{1}{2}d(u, fu) \leq d(u, v) \quad \text{implies} \quad d(fu, fv) \leq d(u, v).$$

We begin with the following definition and lemmas which are useful in proving our result.

Definition 1.2. [14] Let Ψ denote the class of function $\psi : [0, \infty) \to [0, \infty)$ (called altering distance function), which satisfies the following assumptions:

(Ψ1.) ψ is non-decreasing and continuous,
(Ψ2.) $\psi(\omega) = 0$ if and only if $\omega = 0$.

Lemma 1.1. [5] *Let $\pi : [0, \infty) \to [0, \infty)$ is a continuous function. If ψ is an altering distance function satisfying condition $\psi(\omega) > \pi(\omega)$ for all $\omega > 0$, then $\pi(0) = 0$.*

Lemma 1.2. [6] *Let (Y, d) be a metric space. Let $\{u_n\}$ be a sequence in Y such that*

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0.$$

If $\{u_n\}$ is not a Cauchy sequence in Y then there exist an $\epsilon > 0$ and sequences of positive integers (m_k) and (n_k) with $m_k > n_k > k$ such that

$$d(u_{m_k}, u_{n_k}) \geq \epsilon, \quad d(u_{m_k-1}, u_{n_k}) < \epsilon$$

and

(B1.) $\lim_{k\to\infty} d(u_{m_k-1}, u_{n_k+1}) = \epsilon$,
(B2.) $\lim_{k\to\infty} d(u_{m_k}, u_{n_k}) = \epsilon$,
(B3.) $\lim_{k\to\infty} d(u_{m_k-1}, u_{n_k}) = \epsilon$.

In this paper, we first define a generalized $C_\beta^\psi-$ rational contraction and then prove the existence and uniqueness of fixed points for self monotone map. We also consider a partially ordered set Y with comparable elements, and a complete metric d with set $Y$to deduce our main result. As application, we give an existence as well as uniqueness theorem for ordinary differential equation of second order and first degree with restricted boundary conditions.

2. FIXED POINT RESULT WITH PARTIAL ORDER

We define generalized $C_\beta^\psi-$ rational contraction as follows:

Definition 2.1. A mapping f on a metric space (Y, d) is said to satisfy generalized $C_\beta^\psi-$ rational contraction if, for all $u, v \in Y$,

$$\frac{1}{2}d(u, fu) \leq d(u, v) \quad \text{implies} \quad \psi(d(fu, fv)) \leq \beta(M(u, v)), \tag{1}$$

where

$$(2) \qquad M(u,v) = \max\left\{ d(u,v), \frac{d(u,fu)d(v,fv)}{[1+d(u,v)]}, \frac{d(v,fv)[1+d(u,fu)]}{[1+d(u,v)]} \right\},$$

$\beta : [0,\infty) \to [0,\infty)$ is continuous function and $\psi \in \Psi$.

Main finding of this article is the following result.

Theorem 2.1. *Let $(Y, d, \preceq)$ be a partially ordered complete metric space and let $f : Y \to Y$ be a non-decreasing, monotone map satisfying generalized $C_\beta^\psi-$ rational contraction. Also, suppose $\beta : [0,\infty) \to [0,\infty)$ is continuous function and $\psi \in \Psi$ satisfying*

$$(3) \qquad 0 < \beta(\omega) < \psi(\omega), \quad \omega > 0.$$

Also assume that:

(4) *For every $u, v \in Y$, there exists $z \in Y$, such that $u \preceq z$ and $v \preceq z$.*

If there exists $u_0 \in Y$ such that $u_0 \preceq fu_0$, then f has a unique fixed point in Y.

Proof. Let $u_0 \in Y$ satisfy $u_0 \preceq fu_0$. We define a sequence $\{u_n\}$ as follows:

$$(5) \qquad u_n = fu_{n-1}, \; n \in N.$$

If $u_n = u_{n+1}$ for some $n \in N$, then, clearly $M(u_n, u_{n+1}) = 0$ and so, u_n is the fixed point of f. So, assume that $u_n \neq u_{n+1}$ for all $n \in N$. Let $a_n = d(u_n, u_{n+1})$. Then, clearly $a_n > 0$. Since $u_0 \preceq fu_0 = u_1$ and f is non-decreasing, then

$$(6) \qquad u_0 \preceq u_1 \preceq u_2 \cdots \preceq u_n \cdots.$$

On taking $u = u_n$ and $v = fu_n = u_{n+1}$ in (1), we obtain that

$$\frac{1}{2}d(u_n, fu_n) = \frac{1}{2}d(u_n, u_{n+1}) \leq d(u_n, u_{n+1})$$

implies

$$(7) \qquad \psi(d(fu_n, fu_{n+1})) = \psi(d(u_{n+1}, u_{n+2})) \leq \beta(M(u_n, u_{n+1})),$$

where

$$M(u_n, u_{n+1}) = \max\left\{ \begin{array}{c} d(u_n, u_{n+1}), \frac{d(u_n, fu_n)d(u_{n+1}, fu_{n+1})}{[1+d(u_n, u_{n+1})]}, \\ \frac{d(u_{n+1}, fu_{n+1})[1+d(u_n, fu_n)]}{[1+d(u_n, u_{n+1})]} \end{array} \right\}$$

$$= \max\left\{ \begin{array}{c} d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), \\ \frac{d(u_n, u_{n+1})d(u_{n+1}, u_{n+2})}{[1+d(u_n, u_{n+1})]} \end{array} \right\}.$$

Since $\frac{d(u_n, u_{n+1})}{[1+d(u_n, u_{n+1})]} < 1$ for all $n \in \mathbb{N}$, therefore

$$\frac{d(u_n, u_{n+1})d(u_{n+1}, u_{n+2})}{[1+d(u_n, u_{n+1})]} < d(u_{n+1}, u_{n+2}),$$

and hence

$$M(u_n,u_{n+1}) \leq \max\{d(u_n,u_{n+1}), d(u_{n+1},u_{n+2})\}.$$

From (7), we have

$$\psi(d(u_{n+1},u_{n+2})) \leq \beta(\max\{d(u_n,u_{n+1}), d(u_{n+1},u_{n+2})\}). \tag{8}$$

If $d(u_n,u_{n+1}) < d(u_{n+1},u_{n+2})$, then (8) gives a contradiction to condition (3) and hence

$$\psi(d(u_{n+1},u_{n+2})) \leq \beta(d(u_n,u_{n+1})).$$

Since ψ and β are continuous functions, therefore

$$d(u_{n+1},u_{n+2}) \leq d(u_n,u_{n+1}).$$

Similarly we get

$$d(u_n,u_{n+1}) \leq d(u_{n-1},u_n).$$

Thus, we get a sequence $\{d(u_n,u_{n+1})\}$ of functions, which is non-increasing and $r \geq 0$ such that

$$\lim_{n\to\infty} d(u_n,u_{n+1}) = r. \tag{9}$$

However, by taking $\lim_{n\to\infty}$ on both side of (8), we get $\psi(r) \leq \beta(r)$, which is a contradiction to (2). Thus we have $r = 0$, and hence

$$\lim_{n\to\infty} d(u_n,u_{n+1}) = r = 0. \tag{10}$$

Assume on contrary that sequence $\{u_n\}$ is not Cauchy. Then for every $\epsilon > 0$, we can find subsequences of positive integers m_k and n_k, where $n_k > m_k > k$, for all $k \in N$, such that

$$d(u_{m_k},u_{n_k}) > \epsilon \quad \text{and} \quad d(u_{m_k},u_{n_{k-1}}) \leq \epsilon. \tag{11}$$

Also for this $\epsilon > 0$, the convergence of sequence $\{d(u_n,u_{n+1})\}$ implies, there exists $N_0 \in N$ such that $d(u_n,u_{n+1}) < \epsilon$ for all $n \geq N_0$. Let $N_1 = \max\{m_i, N_0\}$. Then, for all $m_k > n_k \geq N_1$, we have

$$d(u_{n_k},u_{n_k+1}) < \epsilon \leq d(u_{n_k},u_{m_k}),$$

where $m_k > n_k$ and hence

$$\frac{1}{2}d(u_{n_k},u_{n_k+1}) \leq d(u_{n_k},u_{m_k}).$$

Now from (1), on substituting $u = u_{n_k}$ and $v = u_{m_k}$, we get

$$\psi(d(fu_{n_k},fu_{m_k})) = \psi(d(u_{n_k+1},u_{m_k+1})) \leq \beta(M(u_{n_k},u_{m_k})) \tag{12}$$

where,

$$M(u_{n_k},u_{m_k}) = \max\left\{\begin{array}{c} d(u_{n_k},u_{m_k}), \frac{d(u_{n_k},fu_{n_k})d(u_{m_k},fu_{m_k})}{[1+d(u_{n_k},u_{m_k})]}, \\ \frac{d(u_{m_k},fu_{m_k})[1+d(u_{n_k},fu_{n_k})]}{[1+d(u_{n_k},u_{m_k})]} \end{array}\right\}$$

$$= \max \left\{ \begin{array}{c} d(u_{n_k}, u_{m_k}), \frac{d(u_{n_k}, u_{n_{k+1}}) d(u_{m_k}, u_{m_{k+1}})}{[1+d(u_{n_k}, u_{m_k})]}, \\ \frac{d(u_{m_k}, u_{m_{k+1}})[1+d(u_{n_k}, u_{n_{k+1}})]}{[1+d(u_{n_k}, u_{m_k})]} \end{array} \right\}. \tag{13}$$

On using Lemma 1.2 and letting $k \to \infty$ in (12) and (13), we obtain $\psi(\epsilon) \leq \beta(\epsilon)$, that's a contradiction to (3) and hence by Lemma 1.1, we get $\epsilon = 0$. This contradicts the assumption that $\epsilon > 0$. Therefore our assumption is wrong. Hence $\{u_n\}$ is Cauchy. Since Y is complete, so $\{u_n\}$ converges with all its subsequences to some limiting value, say $z \in Y$.
Now assume for every $n \in N$

$$d(u_n, z) < \frac{1}{2} d(u_n, u_{n+1})$$

and

$$d(u_{n+1}, z) < \frac{1}{2} d(u_{n+1}, u_{n+2}).$$

Then we have

$$\begin{aligned} d(u_n, u_{n+1}) &\leq d(u_n, z) + d(u_{n+1}, z) \\ &< \frac{1}{2} \Big[d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) \Big] \\ &\leq d(u_n, u_{n+1}), \end{aligned}$$

this is a contradiction. Hence we must have $d(u_n, z) \geq \frac{1}{2} d(u_n, u_{n+1})$ or $d(u_{n+1}, z) \geq \frac{1}{2} d(u_{n+1}, u_{n+2})$, for all $n \in N$. Thus for a sub-sequence $\{n_k\}$ of N, we obatin

$$\frac{1}{2} d(u_{n_k}, f u_{n_k}) = \frac{1}{2} d(u_{n_k}, u_{n_k+1}) \leq d(u_{n_k}, z), \; k \in N,$$

which implies

$$\psi(d(f u_{n_k}, f z)) = \beta(M(u_{n_k}, z)), \tag{14}$$

where

$$M(u_{n_k}, z) = \max \left\{ \begin{array}{c} d(u_{n_k}, z), \frac{d(u_{n_k}, f z) d(u_{n_k}, f u_{n_k})}{[1+d(u_{n_k}, z)]}, \\ \frac{d(z, f z)[1+d(u_{n_k}, f u_{n_k})]}{[1+d(u_{n_k}, z)]} \end{array} \right\}. \tag{15}$$

Both, on letting $k \to \infty$, and using (15) in (14), we get

$$\psi(d(z, f z)) \leq \beta(d(z, f z)).$$

Lemma 1.1 implies that $d(z, f z) = 0$. That is, $f z = z$.
To establish uniqueness, we suppose on contradictory that for all $u, v \in Y$, $u = f u$ and $v = f v$ provided $u \neq v$. Now we discuss following two case for both elements.

Case 1. Without loss of generality, suppose that $u \preceq v$ are comparable. Then

$$0 = \frac{1}{2}d(u, fu) \leq d(u, v),$$

implies that

$$\psi(d(fu, fv)) = \psi(d(u, v)) \leq \beta(M(u, v)) = \beta(d(u, v)), \tag{16}$$

Thus from (2) and Lemma 1.1, we get $d(u, v) = 0$, *i.e,* $u = v$.

Case 2. Assume that u and v are not comparable then from (4), there exists some $z \in Y$ comparable to u and v such that $fz = z$ is comparable $u = fu$ and $v = fv$.

Clearly,

$$0 = d(u, u) = \frac{1}{2}d(u, fu) < d(u, w)$$

implies that

$$\psi(d(fu, fw) \leq \beta(M(u, w)), \tag{17}$$

where

$$\begin{aligned} M(u, w) &= \max\left\{d(u, w), \frac{d(u, fu)d(w, fw)}{1 + d(u, w)}, \frac{d(w, fw)[1 + d(u, fu)]}{1 + d(u, w)}\right\} \\ &= \max d(u, w), 0, 0 = d(u, w). \end{aligned}$$

Hence, from (17),

$$\psi(d(fu, fw) \leq \beta(d(u, w)).$$

Consequently, we have

$$\psi(d(u, w) \leq \beta(d(u, w)).$$

On using Lemma 1.1, we have $d(u, w) = 0$.
Similarly, we can obtain $d(v, w) = 0$. This implies that $u = v$.
This completes the proof of Theorem 2.1.

□

Theorem 2.2. *Let $(Y, d, \preceq)$ be a partially ordered complete metric space and let $f : Y \to Y$ be a non-decreasing, monotone map such that for all $u, v \in Y$,*

$$\frac{1}{2}d(u, fu) \leq d(u, v) \quad \textit{implies} \quad \psi(d(fu, fv)) \leq \beta(N(u, v)), \tag{18}$$

and

$$N(u, v) = a_1 d(u, v) + a_2 \frac{d(u, fu)d(v, fv)}{[1 + d(u, v)]} + a_3 \frac{d(v, fv)[1 + d(u, fu)]}{[1 + d(u, v)]}, \tag{19}$$

where $\psi \in \Psi$, $a_i \geq 0$, $\sum a_i < 1$, for all $i = 1, 2, 3$ and $\beta : [0, \infty) \to [0, \infty)$ is continuous function such that

$$0 < \beta(\omega) < \psi(\omega), \quad \omega > 0. \tag{20}$$

Also assume that, for every $u, v \in Y$, there exists $z \in Y$, such that $u \preceq z$ and $v \preceq z$. If there exists $u_0 \in Y$ such that $u_0 \preceq fu_0$, then f has a unique fixed point in Y.

Proof. Given that $f : Y \to Y$ be monotone, nondecreasing map such that for all $u, v \in Y$,

$$\frac{1}{2}d(u, fu) \leq d(u, v) \quad \text{implies} \quad \psi(d(fu, fv)) \leq \beta(N(u, v)),$$

and

$$\begin{aligned} N(u,v) &= a_1 d(u,v) + a_2 \frac{d(u,fu)d(v,fv)}{[1+d(u,v)]} + a_3 \frac{d(v,fv)[1+d(u,fu)]}{[1+d(u,v)]} \\ &= \sum a_i . \max\left\{ d(u,v), \frac{d(u,fu)d(v,fv)}{[1+d(u,v)]}, \frac{d(v,fv)[1+d(u,fu)]}{[1+d(u,v)]} \right\}. \end{aligned}$$

Since all $a_i \geq 0$ and $\sum a_i < 1$, for all $i = 1, 2, 3$, then

$$\begin{aligned} N(u,v) &\leq \max\left\{ d(u,v), \frac{d(u,fu)d(v,fv)}{[1+d(u,v)]}, \frac{d(v,fv)[1+d(u,fu)]}{[1+d(u,v)]} \right\} \\ &= M(u,v). \end{aligned}$$

Rest of the proof follows directly from main result (Theorem 2.1). □

If we take $a_2 = a_3 = 0, a_1 = 1$ in Theorem 2.2, we obtain following result of Yan et al. [5] satisfying weaker type of C_β^ψ- condition.

Corollary 2.1. *Let $(Y, d, \preceq)$ be a partially ordered complete metric space and let $f : Y \to Y$ be a non-decreasing map such that for all $u, v \in Y$,*

$$\frac{1}{2}d(u, fu) \leq d(u, v) \quad \textit{implies} \quad \psi(d(fu, fv)) \leq \beta(d(u, v)),$$

where $\psi \in \Psi$ and $\beta : [0, \infty) \to [0, \infty)$ is a continuous function such that

$$0 < \beta(\omega) < \psi(\omega), \quad \omega > 0.$$

Also assume that for every $u, v \in Y$, there exists $z \in Y$, such that $u \preceq z$ and $v \preceq z$. If there exists $u_0 \in Y$ such that $u_0 \preceq fu_0$, then f has a unique fixed point in Y.

If we take $\psi(\omega) = \omega$ and $\beta(\omega) = \omega$ in Theorem 2.2, we get the following new result.

Corollary 2.2. *Let $(Y, d, \preceq)$ be a partially ordered complete metric space and let $f : Y \to Y$ be a non-decreasing map such that for all $u, v \in Y$,*

$$\frac{1}{2}d(u, fu) \leq d(u, v) \quad \textit{implies} \quad d(fu, fv) \leq N(u, v),$$

and

$$N(u,v) = a_1 d(u,v) + a_2 \frac{d(u,fu)(v,fv)}{[1+d(u,v)]} + a_3 \frac{d(v,fv)[1+d(u,fu)]}{[1+d(u,v)]},$$

where $a_i \geq 0$, $\sum a_i < 1$, *for all* $i = 1, 2, 3$. *Also assume that for every* $u, v \in Y$, *there exists* $z \in Y$, *such that* $u \preceq z$ *and* $v \preceq z$. *If there exists* $u_0 \in Y$ *such that* $u_0 \preceq fu_0$, *then* f *has a unique fixed point in* Y.

Remark 2.1. If we take $a_2 = 0$ in Corollary 2.2, we obtain the result of Dass and Gupta [24] in fame work of partially ordered metric spaces satisfying C-condition.

Remark 2.2. If we take $a_1 = 0$ and $a_2 = 0$ in Corollary 2.2, we get new result in the sense of partially ordered metric spaces satisfying C- condition.

3. APPLICATION: EXISTENCE OF SOLUTION OF SECOND ORDER BOUNDARY VALUE PROBLEM

We consider following second order differential equation with boundary condition

$$
\begin{aligned}
-\frac{d^2u}{d\omega^2} &= f(\omega, u(\omega)), \quad \omega \in L = [0,1],\ u \in [0, \infty), \\
u(0) &= u'(1) = 0.
\end{aligned} \tag{21}
$$

If $u \in C^2(L)$ is zero of (21), then $u \in C(L)$ is also a zero of following integral equation

$$
u(\omega) = \int_0^T G(\omega, \theta) f(\theta, u(\theta)) d\theta \quad \text{for all } \omega \in L,
$$

where $G(\omega, \theta)$ is the Green function given by

$$
G(\omega, \theta) = \begin{cases} \omega, & \text{if } 0 \leq \omega \leq \theta \leq 1, \\ \theta, & \text{if } 0 \leq \theta \leq \omega \leq 1. \end{cases}
$$

Theorem 3.1. *Consider a second order differential equation (21) with a map* $f : L \times R \to R$. *Assume that* f *is weakly increasing with respect to second variable and continuous. If there exist* $\lambda \in (0, 2]$ *such that*

$$
f(\omega, u) - f(\omega, v) \leq \lambda \sqrt{\log[(u-v)^2 + 1]},\ u \geq v,
$$

then there exist a unique non negative solution for the problem (21).

Proof. If we let $S = \{u \in C(L), L = [0,1] : u(\omega) \geq 0\}$ be a cone, and (S, d) be a metric space with metric defined as $d(u, v) = sup\{|u(\omega) - v(\omega)| : \omega \in L\}$; for all $u, v \in E$, then clearly (S, d) is complete.
Define $H : C(L) \to C(L)$ by

$$
(Hu)(\omega) = \int_0^1 G(\omega, \theta) f(\theta, u(\theta)) d\theta.
$$

If $u \in C(L)$ is a fixed point of H, then $u \in C^1(L)$ is a zero of (21).
Clearly, with assumption on f and elements $u, v \in E$, we obtain

$$
(Hu)(\omega) = \int_0^1 G(\omega, \theta) f(\theta, u(\theta)) d\theta \geq \int_0^1 G(\omega, \theta) f(\theta, v(\theta)) d\theta = (Hv)(\omega).
$$

Since $G(\omega,\theta)>0$, for $\omega\in L$. This proves that H is also weakly increasing mapping.

Also, for all $u,v\in E$ with $u\geq v$ implies that

$$\sup\{|u(\omega)-v(\omega)|,\omega\in L\}\geq\sup\{|Hu(\omega)-u(\omega)|,\omega\in L\},\tag{22}$$

and so, in term of metric

$$d(u,v)\geq d(Hu,u)\geq\frac{1}{2}d(Hu,u).\tag{23}$$

This implies

$$\begin{aligned}
d(Hu,Hv)&=\sup_{\omega\in L}|(Hu)(\omega)-(Hv)(\omega)|=\sup_{\omega\in L}((Hu)(\omega)-(Hv)(\omega))\\
&=\sup_{\omega\in L}\int_0^1 G(\omega,\theta)[f(\theta,u(\theta))-f(\theta,v(\theta))]d\theta\\
&=\sup_{\omega\in L}\int_0^1 G(\omega,\theta)\lambda\sqrt{\log[(u(\theta)-v(\theta))^2+1]}d\theta\\
&=\sup_{\omega\in L}\int_0^1 G(\omega,\theta)\lambda\sqrt{\log[d(u,v)^2+1]}d\theta\\
&=\lambda\sqrt{\log[d(u,v)^2+1]}\sup_{\omega\in L}\int_0^1 G(\omega,\theta)d\theta. \qquad (24)
\end{aligned}$$

It is easy to calculate that

$$\int_0^1 G(\omega,\theta)d\theta=\frac{-\omega^2}{2}+\omega,$$

and so

$$\sup_{\omega\in L}\int_0^1 G(\omega,\theta)d\theta=\frac{1}{2}.\tag{25}$$

On using (25) in (24), we get

$$d(Hu,Hv)\leq\frac{\lambda}{2}\sqrt{\log[d(u,v)^2+1]}.\tag{26}$$

Since, $\lambda\in(0,2]$, we obatin

$$d(Hu,Hv)\leq\sqrt{\log[d(u,v)^2+1]},$$

and that

$$d(Hu,Hv)^2\leq\log[d(u,v)^2+1].\tag{27}$$

Assuming $\psi(\omega)=\omega^2$ and $\beta(\omega)=\log[\omega^2+1]$. Then clearly, $\psi\in\Psi$, and for all $u>0$, $\psi(u)>\beta(u)$.
Relation (27) implies that

$$\psi(d(Hu,Hv))\leq\beta(d(u,v))$$

$$\leq \beta\left(\max\left\{ d(u,v), \frac{d(u,fu)d(v,fv)}{[1+d(u,v)]}, \frac{d(v,fv)[1+d(u,fu)]}{[1+d(u,v)]} \right\} \right)$$
$$= \beta(M(u,v)).$$

Also,

$$H(0) = \int_0^1 G(\omega,\theta) f(\theta,0) d\theta \geq 0.$$

Thus one by one all assumptions of Theorem 2.1 are satisfied and therefore, the function H has a unique non negative solution. □

4. Conclusion

In this manuscript, we have first defined a generalized $C_\beta^\psi-$ rational contraction and then derived our main result Theorem 2.1. Some consequence results (Corollary 2.1, 2.2) and Remarks 2.1, 2.2 flaunted that our result is a proper generalization and extension of some previous existing results. As an application of our main result, we have presented an example to find the existence and uniqueness of solutions of second order boundary value problem.

5. Acknowledgments

We wish to confirm that there has been no significant financial support for this work that could have influenced its outcome and the authors declare that they have no conflict of interest.

References

[1] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., 132 (5) (2004), 1435–1443.

[2] A. Amini Harandi, H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Anal., 72 (5) (2010), 2238–2242.

[3] E. Karapinar, K. Tas, *Generalized (C)-conditions and related fixed point theorems*, Comput. Math. Appl., 61 (11) (2011), 3370–3380.

[4] E. Karapinar, I. M. Erhan, U. Aksoy, *Weak ψ - contractions on partially ordered metric spaces and applications to boundary value problems*, Boundary Value Problems, 2014 (2014), Article ID: 149.

[5] F. Yan, Y. Su, Q. Feng, *A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations*, Fixed Point Theory Appl., 2012 (2012), Article ID: 152.

[6] G. V. R. Babu, P. D. Sailaja, *A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces*, Thai Jour. Math., 9 (1) (2011), 1–10.

[7] H. K. Nashine, B. Samet, *Fixed point results for mappings satisfying* $(\psi, \varphi)-$ *weakly contractive condition in partially ordered metric spaces*, Nonlinear Anal., 74 (6) (2011), 2201–2209.

[8] I. Altun, H. Simsek, *Some fixed point theorems on ordered metric spaces and application*, Fixed Point Theory Appl., (2010) (2010), Article ID: 621469.

[9] J. J. Nieto, R. R. Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, 22 (3) (2005), 223–239.

[10] J. J. Nieto, R. R. Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta. Math. Sin.-English Ser., 23 (2007), 2205–2212.

[11] J. J. Nieto, R. L. Pouso, R. R. Lopez *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc., 135 (8) (2007), 2505–2517.

[12] J. Harjani, K. Sadarangni, *Fixed point theorems for weakly contraction mappings in partially ordered sets*, Nonlinear Anal., 71 (2009), 3403–3410.

[13] J. Harjani, K. Sadarangni *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*, Nonlinear Anal., 72 (2010), 1188–1197.

[14] M. S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc., 30 (1984), 1–9.

[15] T. G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., 65 (2006), 1379–1393.

[16] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, Journal of mathematical analysis and applications, 340 (2008), 1088–1095.

[17] T. Suzuki, *A new type of fixed point theorem on metric spaces*, Nonlinear Anal., 71 (2009), 5313–5317.

[18] V. Lakshmikantham, Lj. B. Ciric, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., 70 (2009), 4341–4349.

[19] V. Gupta, W. Shatanawi, N. Mani, *Fixed point theorems for* (ψ, β) *-Geraghty contraction type maps in ordered metric spaces and some applications to integral and ordinary differential equations*, J. Fixed Point Theory Appl., 19 (2) (2016), 1251–1267.

[20] V. Gupta, N. Mani *Existence and uniqueness of fixed point for contractive mapping of integral type*, International Journal of Computing Science and Mathematics, 4 (1) (2013), 72–83.

[21] V. Gupta, Ramandeep, N. Mani, A. K. Tripathi, *Some fixed point result involving generalized altering distance function*, Procedia Computer Science, 79 (2016), 112–117.

[22] W. Shatanawi, A. Al Rawashdeh, *Common fixed points of almost generalized* (ψ, ϕ)*-contractive mappings in ordered metric spaces*, Fixed Point Theory Appl., 2012 (2012), Article ID: 80.

[23] D. S. Jaggi, *Some unique fixed point theorems*, Indian J. Pure Appl. Math., 8s (1977), 223–230.

[24] B. K. Dass, S. Gupta, *An extension of Banach contraction principle through rational expression*, Indian J. Pure Appl. Math., 12 (6) (1975), 1455–1458.

[25] V. K. Bhardwaj, V. Gupta, N. Mani, *Common fixed point theorems without continuity and compatible property of maps*, Bol. Soc. Paran. Mat., 35 (3) (2017), 67–77.

NAVEEN MANI
DEPARTMENT OF MATHEMATICS
SANDIP UNIVERSITY, NASHIK
MAHARASHTRA
INDIA
E-mail address: naveenmani81@gmail.com