Relation between \(b\)-metric and fuzzy metric spaces

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ABSTRACT. In this work we have considered several common fixed point results in \(b\)-metric spaces for weak compatible mappings. By applications of these results we establish some fixed point theorems in \(b\)-fuzzy metric spaces.

1. Introduction

In this paper we establish some fixed point results in a \(b\)-fuzzy metric space by applications of certain fixed point theorems in \(b\)-metric spaces. Also we prove some fixed point results in \(b\)-metric spaces. Fuzzy metric space was first introduced by Kramosil and Michalek [3]. Subsequently, George and Veeramani had given a modified definition of fuzzy metric spaces [1]. Fixed point results in such spaces have been established in a large number of works. Some of these works are noted in [2, 4, 5, 7, 10, 11].

Definition 1.1. [1] A binary operation \(* : [0, 1] \times [0, 1] \rightarrow [0, 1]\) is a continuous \(t\)-norm if it satisfies the following conditions:

1. \(*\) is associative and commutative,
2. \(*\) is continuous,
3. \(a \ast 1 = a,\) for all \(a \in [0, 1],\)
4. \(a \ast b \leq c \ast d\) whenever \(a \leq c\) and \(b \leq d,\) for each \(a, b, c, d \in [0, 1].\)

Two typical examples of continuous \(t\)-norm are \(a \ast b = ab\) and \(a \ast b = \min(a, b).\)

Definition 1.2. [1] A 3-tuple \((X, M, *)\) is called a fuzzy metric space if \(X\) is an arbitrary (non-empty) set, \(*\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X^2 \times (0, \infty),\) satisfying the following conditions, for each \(x, y, z \in X\) and \(t, s > 0:"

1. \(M(x, y, t) > 0,\)

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(2) \( M(x, y, t) = 1 \) if and only if \( x = y \),
(3) \( M(x, y, t) = M(y, x, t) \),
(4) \( M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \),
(5) \( M(x, y, .) : (0, \infty) \to [0, 1] \) is continuous.

**Definition 1.3.** [8, 9] A 3-tuple \((X, M, *)\) is called a \( b \)-fuzzy metric space if \( X \) is an arbitrary (non-empty) set, * is a continuous \( t \)-norm and \( M \) is a fuzzy set on \( X^2 \times (0, \infty) \), satisfying the following conditions, for each \( x, y, z \in X \), \( t, s > 0 \) and a given real number \( b \geq 1 \):

1. \( M(x, y, t) > 0 \),
2. \( M(x, y, t) = 1 \) if and only if \( x = y \),
3. \( M(x, y, t) = M(y, x, t) \),
4. \( M(x, y, t) \cdot t^b * M(y, z, s) \leq M(x, z, t + s) \),
5. \( M(x, y, .) : (0, \infty) \to [0, 1] \) is continuous.

We present an example shows that a \( b \)-fuzzy metric on \( X \) need not be a fuzzy metric on \( X \).

**Example 1.4.** Let \( M(x, y, t) = e^{-|x-y|^p / t} \), where \( p > 1 \) is a real number. We show that \( M \) is a \( b \)-fuzzy metric with \( b = 2^{p-1} \).

Obviously conditions (1), (2), (3) and (5) of Definition 1.3 are satisfied.

If \( 1 < p < \infty \), then the convexity of the function \( f(x) = x^p \) \( (x > 0) \) implies

\[
\left( \frac{a + c}{2} \right)^p \leq \frac{1}{2} (a^p + c^p),
\]

and hence, \((a + c)^p \leq 2^{p-1}(a^p + c^p)\) holds. Therefore,

\[
\frac{|x - y|^p}{t + s} \leq 2^{p-1} \left( \frac{|x - z|^p}{t + s} + \frac{|z - y|^p}{t + s} \right) \leq 2^{p-1} \frac{|x - z|^p}{t} + \frac{|z - y|^p}{s} \leq \frac{|x - z|^p}{t/2^{p-1}} + \frac{|z - y|^p}{s/2^{p-1}}.
\]

Thus for each \( x, y, z \in X \) we obtain

\[
M(x, y, t + s) = e^{-|x-y|^p / t+s} \geq M(x, z, \frac{t}{2^{p-1}}) * M(z, y, \frac{s}{2^{p-1}}),
\]

where \( a * b = ab \). So condition (4) of Definition 1.3 hold and \( M \) is a \( b \)-fuzzy metric.

It should be noted that in preceding example, for \( p = 2 \) it is easy to see that \((X, M, *)\) is not a fuzzy metric space.
Example 1.5. Let $M(x, y, t) = e^{-d(x, y)t}$ or $M(x, y, t) = \frac{t}{t + d(x, y)}$, where $d$ is a $b$-metric on $X$ and $a * c = ac$, for all $a, c \in [0, 1]$. Then it is easy to show that $M$ is a $b$-fuzzy metric.

Obviously conditions (1), (2), (3) and (5) of Definition 1.3 are satisfied. For each $x, y, z \in X$ we obtain

$$M(x, y, t + s) = e^{-d(x, y)t + s} \geq e^{-d(x, z)d(x, y)t + s} \cdot e^{-d(x, y)s} \geq M(x, z, \frac{t}{2b}) \cdot M(z, y, \frac{s}{2b}) = M(x, z, \frac{t}{b}) * M(z, y, \frac{s}{b}).$$

So condition (4) of Definition 1.3 is hold and $M$ is a $b$-fuzzy metric. Similarly, it is easy to see that $M(x, y, t) = \frac{t}{t + d(x, y)}$ is a $b$-fuzzy metric.

2. MAIN RESULTS

Lemma 2.1. Let $(X, M, *)$ be a $b$-fuzzy metric space with $a * c \geq ac$, for all $a, c \in [0, 1]$. If $d : X^2 \to [0, \infty)$ is defined by $d(x, y) = \lim_{\epsilon \to 0} \int_0^1 \log_\alpha M(x, y, t)dt$, for $0 < \alpha < 1$, then $d$ is an $2b$-metric on $X$.

Proof. By definition, we have that $d(x, y)$ is well defined for each $x, y \in X$. Clearly, $d(x, y) \geq 0$, for all $x, y \in X$. Moreover, $d(x, y) = 0$ if and only if $\log_\alpha M(x, y, t) = 0$ if and only if $M(x, y, t) = 1$ if and only if $x = y$.

Since

$$M(x, y, t) \geq M(x, z, \frac{t}{2b}) * M(z, y, \frac{t}{2b}) \geq M(x, z, \frac{t}{2b}) \cdot M(z, y, \frac{t}{2b}),$$

it follows that

$$d(x, y) = \lim_{\epsilon \to 0} \int_0^1 \log_\alpha M(x, y, t)dt \leq \lim_{\epsilon \to 0} \int_0^1 \log_\alpha M(x, z, \frac{t}{2b}) \cdot M(z, y, \frac{t}{2b})dt \leq \lim_{\epsilon \to 0} \int_0^1 \log_\alpha M(x, z, \frac{t}{2b})dt + \lim_{\epsilon \to 0} \int_0^1 \log_\alpha M(z, y, \frac{t}{2b})dt = 2blim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^1 \log_\alpha M(x, z, t)dt + 2blim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^1 \log_\alpha M(z, y, t)dt \leq 2b \left[ \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^1 \log_\alpha M(x, z, t)dt + \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^1 \log_\alpha M(x, z, t)dt \right].$$
Relation between $b$-metric and fuzzy metric spaces

\[ 2b[d(x, z) + d(z, y)]. \]

This proves that $d$ is an $2b$-metric on $X$. $\square$

The following lemma plays an important role to give fixed point results on a fuzzy metric space.

Lemma 2.2. Let $(X, M, \ast)$ be a $b$-fuzzy metric space with $a \ast c \geq ac$, for all $a, c \in [0, 1]$. If $d : X^2 \rightarrow [0, \infty)$ is define by $d(x, y) = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(x, y, t) dt$, for all $0 < \alpha < 1$, then:

1. $\{x_n\}$ is a Cauchy sequence in $b$-fuzzy metric $(X, M, \ast)$ if and only if it is a Cauchy sequence in the $2b$-metric space $(X, d)$.
2. A $b$-fuzzy metric space $(X, M, \ast)$ is complete if and only if the $2b$-metric space $(X, d)$ is complete.

Proof. First we show that every Cauchy sequence in $(X, M, \ast)$ is a Cauchy sequence in $(X, d)$. To this end let $\{x_n\}$ be a Cauchy sequence in $(X, M, \ast)$. Then $\lim_{n,m \to \infty} M(x_n, x_m, t) = 1$. Since $d(x_n, x_m) = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(x_n, x_m, t) dt$, is a $2b$-metric. Hence, we have

\[
\lim_{n,m \to \infty} d(x_n, x_m) = \lim_{n,m \to \infty} \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(x_n, x_m, t) dt = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} \lim_{n,m \to \infty} M(x_n, x_m, t) dt = 0,
\]

so, we conclude that $\{x_n\}$ is a Cauchy sequence in $(X, d)$.

Next we prove that completeness of $(X, d)$ implies completeness of $(X, M, \ast)$. Indeed, if $\{x_n\}$ is a Cauchy sequence in $(X, M, \ast)$ then it is also a Cauchy sequence in $(X, d)$. Since the $2b$-metric space $(X, d)$ is complete we deduce that there exists $y \in X$ such that $\lim_{n \to \infty} d(x_n, y) = 0$. Therefore,

\[
\lim_{n \to \infty} \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(x_n, y, t) dt = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} \lim_{n \to \infty} M(x_n, y, t) dt = 0,
\]

that is $\lim_{n \to \infty} M(x_n, y, t) dt = 1$. Hence we follow that $\{x_n\}$ is a convergent sequence in $(X, M, \ast)$.

Now we prove that every Cauchy sequence $\{x_n\}$ in $(X, d)$ is a Cauchy sequence in $(X, M, \ast)$. Therefore,

\[
\lim_{n \to \infty} d(x_n, x_m) = \lim_{n \to \infty} \lim_{m \to \infty} \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(x_n, x_m, t) dt = 0.
\]
Hence, \( \lim_{n,m \to \infty} M(x_n, x_m, t) = 1 \).

That is, \( \{x_n\} \) is a Cauchy sequence in \((X, M, \ast)\).

We will establish the lemma if we prove that \((X, d)\) is complete if so is \((X, M, \ast)\). Let \( \{x_n\} \) be a Cauchy sequence in \((X, d)\). Then \( \{x_n\} \) is a Cauchy sequence in \((X, M, \ast)\), and so it is convergent to a point \( y \in X \) with

\[
\lim_{n \to \infty} M(x_n, y, t) = 1.
\]

As a consequence we have

\[
\lim_{n \to \infty} d(x_n, y) = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(x_n, y, t) dt = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} \lim_{n \to \infty} M(x_n, y, t) dt = 0.
\]

Therefore \((X, d)\) is complete.

\( \square \)

**Lemma 2.3.** Let \((X, M, \ast)\) be a \(b\)-fuzzy metric space with \( a \ast c = \min\{a, c\} \), for all \( a, c \in [0, 1] \). We define \( d : X^2 \to [0, \infty) \) by

\[
d(x, y) = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \cot\left(\frac{\pi}{2} M(x, y, t)\right) dt,
\]

then \( d \) is an \(2b\)-metric on \( X \).

**Proof.** Clearly, \( d(x, y) \geq 0 \), for all \( x, y \in X \). Moreover, \( d(x, y) = 0 \) if and only if \( \cot\left(\frac{\pi}{2} M(x, y, t)\right) = 0 \) if and only if \( M(x, y, t) = 1 \) if and only if \( x = y \).

Since,

\[
M(x, y, t) \geq M(x, z, \frac{t}{2b}) \ast M(z, y, \frac{t}{2b}) = \min\{M(x, z, \frac{t}{2b}), M(z, y, \frac{t}{2b})\},
\]

and also since \( 0 < \frac{\pi}{2} M(x, y, \frac{t}{2b}) \leq \frac{\pi}{2} \), it follows that,

\[
d(x, y) = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \cot\left(\frac{\pi}{2} M(x, y, t)\right) dt
\leq \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \cot\left(\frac{\pi}{2} (M(x, z, \frac{t}{2b}) \ast M(z, y, \frac{t}{2b}))\right) dt
= 2b \left( \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^{\frac{1}{2b}} \cot\left(\frac{\pi}{2} \min\{M(x, z, t), M(z, y, t)\}\right) dt \right)
= 2b \min \left\{ \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^{\frac{1}{2b}} \cot\left(\frac{\pi}{2} M(x, z, t)\right) dt, \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^{\frac{1}{2b}} \cot\left(\frac{\pi}{2} M(z, y, t)\right) dt \right\}
\leq 2b \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^{\frac{1}{2b}} \cot\left(\frac{\pi}{2} M(x, z, t)\right) dt + 2b \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^{\frac{1}{2b}} \cot\left(\frac{\pi}{2} M(z, y, t)\right) dt
= 2b[d(x, z) + d(z, y)],
\]

that is \( d \) is an \(2b\)-metric on \( X \).

\( \square \)
Remark 2.4. Let \( a, b \in (0, 1] \), then it is a standard result that
\[
\arccot(\min\{a, b\}) \leq \arccot(a) + \arccot(b) - \frac{\pi}{4}.
\]

Lemma 2.5. Let \((X, M, \ast)\) be a 2b-fuzzy metric space with \( a \ast c = \min\{a, c\} \), for all \( a, c \in [0, 1] \). If we define \( d : X^2 \to [0, \infty) \) by
\[
d(x, y) = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} (\frac{4}{\pi} \arccot(M(x, y, t)) - 1) dt,
\]
then \( d \) is an 2b-metric on \( X \).

Proof. Clearly, \( 0 \leq d(x, y) < 1 \), for all \( x, y \in X \). Moreover, \( d(x, y) = 0 \) if and only if \( \frac{4}{\pi} \arccot(M(x, y, t)) - 1 = 0 \) if and only if \( \arccot(M(x, y, t)) = \frac{\pi}{4} \) if and only if \( M(x, y, t) = 1 \) if and only if \( x = y \). Since
\[
M(x, y, t) \geq M(x, z, \frac{t}{2b}) \ast M(z, y, \frac{t}{2b}) = \min\{M(x, z, \frac{t}{2b}), M(z, y, \frac{t}{2b})\},
\]
it follows that
\[
\arccot(M(x, y, t)) \leq \arccot[\min\{M(x, z, \frac{t}{2b}), M(z, y, \frac{t}{2b})\}]
= \arccot(\min\{M(x, z, \frac{t}{2b}), M(z, y, \frac{t}{2b})\})
\leq \arccot(M(x, z, \frac{t}{2b})) + \arccot(M(z, y, \frac{t}{2b})) - \frac{\pi}{2}.
\]
Hence,
\[
d(x, y) = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} (\frac{4}{\pi} \arccot(M(x, y, t)) - 1) dt
\leq \lim_{\epsilon \to 0} \int_{\epsilon}^{1} (\frac{4}{\pi} \arccot(M(x, z, \frac{t}{2b})) - 1) dt
+ \lim_{\epsilon \to 0} \int_{\epsilon}^{1} (\frac{4}{\pi} \arccot(M(z, y, \frac{t}{2b})) - 1) dt
= 2b \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^{\frac{1}{2b}} (\frac{4}{\pi} \arccot(M(x, z, t)) - 1) dt
+ 2b \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^{\frac{1}{2b}} (\frac{4}{\pi} \arccot(M(z, y, t)) - 1) dt
\leq 2b \left( \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^{\frac{1}{2b}} (\frac{4}{\pi} \arccot(M(x, z, t)) - 1) dt
+ \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2b}}^{\frac{1}{2b}} (\frac{4}{\pi} \arccot(M(z, y, t)) - 1) dt \right)
= 2b[d(x, z) + d(z, y)],
\]
that is \( d \) is an 2b-metric on \( X \). \( \square \)
Remark 2.6. Let \((X, M, \ast)\) be a fuzzy metric space with \(a \ast c \geq ac\), for all \(a, c \in [0, 1]\). If sequence \(\{x_n\}\) in \(X\) converges to \(x\), that is, for every \(0 < \epsilon < 1\) there exists \(n_0 \in \mathbb{N}\) such that \(M(x_n, x, t) > 1 - \epsilon\), for all \(n \geq n_0\) and each \(t > 0\), then \(d(x_n, x) \to 0\) where \(d(x, y) = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(x, y, t)dt\).

Also it is a Cauchy sequence if for each \(0 < \epsilon < 1\) and \(t > 0\), there exits \(n_0 \in \mathbb{N}\) such that \(M(x_n, x_m, t) > 1 - \epsilon\) for each \(n, m \geq n_0\). It follows that \(d(x_n, x_m) = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(x_n, x_m, t)dt < \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} (1 - \epsilon)dt < \eta\), for every \(\eta = (1 - \alpha) \log_{\alpha} (1 - \epsilon)\). Thus \(\{x_n\}\) in \(2b\)-metric \((X, d)\) is a Cauchy sequence.

Theorem 2.7. \([6]\) Suppose that \(f, g, S\) and \(T\) are self mappings of a complete \(b\)-metric space \((X, d)\), with \(f(X) \subseteq T(X)\), \(g(X) \subseteq S(X)\) and that the pairs \(\{f, S\}\) and \(\{g, T\}\) are compatible. If

\[
(2.1) \quad d(fx, gy) \leq q \frac{1}{4} \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\},
\]

for each \(x, y \in X\), with \(0 < q < 1\). Then \(f, g, S\) and \(T\) have a unique common fixed point in \(X\) provided that \(S\) and \(T\) are continuous.

We next apply theorem 2.7 to establish the following theorem in fuzzy metric spaces.

Theorem 2.8. Let \((X, M, \ast)\) be a complete fuzzy metric space with \(a \ast c \geq ac\) for all \(a, c \in [0, 1]\). Let \(f, g, S\) and \(T\) be self mappings on \(X\) with \(f(X) \subseteq T(X)\), \(g(X) \subseteq S(X)\) and that the pairs \(\{f, S\}\) and \(\{g, T\}\) are compatible. If there exists \(q \in (0, 1)\) such that for each \(x, y \in X\),

\[
M(fx, gy, t) \geq \min\left(\frac{M(Sx, Ty, t), M(fx, Sx, t), M(gy, Ty, t), \sqrt{M(Sx, gy, t) \cdot M(fx, Ty, t)}}{2b}\right)^{\frac{q}{(2b)^{\gamma}}}
\]

If \(S\) and \(T\) are continuous, then \(f, g, S\) and \(T\) have a unique common fixed point in \(X\).

Proof. We define \(d(x, y) = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(x, y, t)dt\) for every \(x, y \in X\) where \(0 < \alpha < 1\). Then by Lemma 2.1 and Lemma 2.2 \((X, d)\) is a complete \(2b\)-metric space. From the above inequality, we get,
\[
\lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(f x, g y, t) dt \leq \frac{q}{(2b)^4} \max \left( \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(S x, T y, t) dt, \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(f x, S x, t) dt, \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(g y, T y, t) dt, \frac{1}{2} \left( \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(S x, g y, t) dt + \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log_{\alpha} M(f x, T y, t) dt \right) \right),
\]
which is,
\[
d(f x, g y) \leq \frac{q}{(2b)^4} \max \left( d(S x, T y), d(f x, S x), d(g y, T y), \frac{1}{2}(d(S x, g y) + d(f x, T y)) \right).
\]
Hence all the conditions of Theorem 2.7 hold, so the conclusion of Theorem 2.8 follows by an application of Theorem 2.7.

\[\square\]

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