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Fixed point theorems of generalised $S - \beta - \psi$ contractive type mappings

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Abstract. In this paper, we introduce the concept of generalised $S - \beta - \psi$ contractive type mappings. For these mappings we prove some fixed point theorems in the setting of S-metric space.

1. Introduction

Banach contraction mapping principle is one of the most important result for finding fixed point. There are various generalisations and extensions of this theorem. One of the important generalisation is the result obtained by Samet et. al [2]. They introduced the concept of α-admissible mapping and defined the notion of $\alpha - \psi$ contractive mappings. Sedghi et. al [9] generalised the concept of metric space to S-metric space. The concept of S-metric is also further extended by many researcher. These generalisations can be seen in [6, 10, 7, 8] and references therein.

In this paper, we extend the notion of $\alpha - \psi$ contractive mapping to $S - \beta - \psi$ contractive type mappings in the setting of S-metric space.

Berinde [1] defined (c)-comparison function as follows:

Let Ψ be a family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions.

(i) ψ is nondecreasing;

(ii) There exists $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of non negative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k,$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$.

Lemma 1.1. [1] *If $\psi \in \Psi$, then the following hold:*

(i) *$(\psi^n(t))_{n \in N}$ converges to 0 as $n \to \infty$ for all $t \in \mathbb{R}^+$;*

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(ii) *$\psi(t) < t$ for any $t \in \mathbb{R}^+$;*
(iii) *ψ is continuous at 0;*
(iv) *the series $\sum_{k=1}^{\infty} \psi^k(t)$ converges for any $t \in \mathbb{R}^+$.*

Samet et. al. [2] defined α-admissible as follows.

Definition 1.1. [2] Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be α-admissible if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Karapinar and Samet [3] introduced the following contractive condition.

Definition 1.2. [3] Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a generalised $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)),$$

for all $x, y \in X$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

The concept of β-admissible mapping is introduced by Alghamdi and Karapinar [4].

Definition 1.3. [4] Let $T : X \to X$ and $\beta : X \times X \times X \to [0, \infty)$, then T is said to be β-admissible if for all $x, y, z \in X$,

$$\beta(x, y, z) \geq 1 \Rightarrow \beta(Tx, Ty, Tz) \geq 1.$$

Example 1.1. [4] Let $X = [0, \infty)$ and $T : X \to X$. Define $\beta(x, y, z) : X \times X \times X \to [0, \infty)$ by $Tx = \log(1 + x)$ and

$$\beta(x, y, z) = \begin{cases} e, & \text{if } x \geq y \geq z, \\ 0, & \text{otherwise.} \end{cases}$$

Then T is called β-admissible.

The definition of S-metric space is as follows.

Definition 1.4. [9] Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions.

(S1) $S(x, y, z) = 0$ if and only if $x = y = z$,
(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$,

for each $x, y, z, a \in X$.
The pair (X, S) is called S-metric space.

Lemma 1.2. [9] *Let (X, S) be an S-metric space. Then*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$$
$$\text{and} \qquad S(x, x, z) \leq 2S(x, x, y) + S(z, z, y),$$

for all $x, y, z \in X$.
Also, $S(x, x, y) = S(y, y, x)$ *for all* $x, y \in X$.

Definition 1.5. [9] Let (X, S) be an S-metric space.

(i) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x) < \epsilon$ whenever $n \geq n_0$ and we denote this $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$.

(iii) The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Definition 1.6. [5] A mapping $T : X \to X$ is said to be S-continuous if $\{Tx_n\}$ is S-convergent to Tx, where $\{x_n\}$ is an S-convergent sequence converging to x.

2. Main results

Now we introduce the concept of generalised $S - \beta - \psi$ contractive mappings by generalising the concept of $\alpha - \psi$ contractive mapping in the setting of S-metric space.

Definition 2.1. Let (X, S) be a S-metric space and let $T : X \to X$ be a given mapping. We say that T is a generalised $S - \beta - \psi$ contractive mapping of type I if there exist two functions $\beta : X \times X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y, z \in X$, we have

$$\beta(x, y, z)S(Tx, Ty, Tz) \leq \psi(M(x, y, z)), \tag{1}$$

where

$$\begin{aligned} M(x, y, z) \;&=\; \max\{S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz), \\ &\quad \frac{1}{4}(S(x, x, Ty) + S(y, y, Tz) + S(z, z, Tx))\}. \end{aligned}$$

Definition 2.2. Let (X, S) be a S-metric space and let $T : X \to X$ be a given mapping. We say that T is a generalised $S-\beta-\psi$ contractive mapping of type II if there exist two functions $\beta : X \times X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have

$$\beta(x, x, y)S(Tx, Tx, Ty) \leq \psi(M(x, x, y)),$$

where

$$\begin{aligned} M(x, x, y) \;&=\; \max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty), \\ &\quad \frac{1}{4}(S(x, x, Tx) + S(x, x, Ty) + S(y, y, Tx))\}. \end{aligned}$$

Theorem 2.1. *Let (X, S) be a complete S-metric space. Suppose that $T : X \to X$ is a generalised $S-\beta-\psi$ contractive mapping of type I and satisfies the following conditions.*

(i) *T is β-admissible;*
(ii) *There exists $x_0 \in X$ such that $\beta(x_0, x_0, Tx_0) \geq 1$;*
(iii) *T is S-continuous.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. Let $x_0 \in X$ be such that $\beta(x_0, x_0, Tx_0) \geq 1$ (such a point exists from the condition (ii)). Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then $u = x_{n_0}$ is a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$ for all n. Since T is β-admissible, we have

$$\beta(x_0, x_0, x_1) = \beta(x_0, x_0, Tx_0) \geq 1 \Rightarrow \beta(Tx_0, Tx_0, Tx_1) = \beta(x_1, x_1, x_2) \geq 1.$$

Inductively, we have

$$\beta(x_n, x_n, x_{n+1}) \geq 1, \tag{2}$$

for all $n = 0, 1, ...$

From (1) and (2), it follows that for all $n \geq 1$, we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq \beta(x_{n-1}, x_{n-1}, x_n)S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq \psi(M(x_{n-1}, x_{n-1}, x_n)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} M(x_{n-1}, x_{n-1}, x_n) &= \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, Tx_{n-1}), \\ &\quad S(x_n, x_n, Tx_n), \frac{1}{4}(S(x_{n-1}, x_{n-1}, Tx_{n-1}) \\ &\quad +S(x_{n-1}, x_{n-1}, Tx_n) + S(x_n, x_n, Tx_{n-1})) \\ &= \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), \\ &\quad S(x_n, x_n, x_{n+1}), \frac{1}{4}(S(x_{n-1}, x_{n-1}, x_n) \\ &\quad +S(x_{n-1}, x_{n-1}, x_{n+1}) + S(x_n, x_n, x_n))\} \\ &\leq \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\quad \frac{1}{4}(S(x_{n-1}, x_{n-1}, x_n) + 2S(x_{n-1}, x_{n-1}, x_n) \\ &\quad +S(x_n, x_n, x_{n+1}))\} \\ &= \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\quad \frac{1}{4}(3S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1}))\} \end{aligned}$$

$$= \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}.$$

Thus, we have

$$S(x_n, x_n, x_{n+1}) \leq \psi(\max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}).$$

We consider the following two cases:

Case I: If $\max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} = S(x_{n-1}, x_{n-1}, x_n)$ for some n, then

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq \psi(S(x_n, x_n, x_{n+1})) \\ &< S(x_n, x_n, x_{n+1}), \end{aligned}$$

which is a contradiction.

Case II: If $\max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} = S(x_{n-1}, x_{n-1}, x_n)$, then

$$S(x_n, x_n, x_{n+1}) \leq \psi(S(x_{n-1}, x_{n-1}, x_n)),$$

for all $n \geq 1$. Since ψ is nondecreasing by induction, we get

$$S(x_n, x_n, x_{n+1}) \leq \psi^n(S(x_0, x_0, x_1), \tag{3}$$

for all $n \geq 1$.

Using $(S2)$ of Definition (1.4) and (3), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\leq 2S(x_n, x_n, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\quad + S(x_{n+2}, x_{n+2}, x_m) \\ &\leq 2\{S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots \\ &\quad \ldots + S(x_{m-2}, x_{m-2}, x_{m-1})\} + S(x_{m-1}, x_{m-1}, x_m) \\ &< 2\{S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots \\ &\quad \ldots + S(x_{m-2}, x_{m-2}, x_{m-1}) + S(x_{m-1}, x_{m-1}, x_m) \\ &= 2\sum_{k=n}^{m-1} S(x_k, x_k, x_{k+1}) \\ &\leq 2\sum_{k=n}^{m-1} \psi^k(S(x_0, x_0, x_1)). \end{aligned}$$

Since $\psi \in \Psi$ and $S(x_0, x_0, x_1) > 0$, by Lemma (1.1), we get that

$$\textstyle\sum_{k=0}^{\infty} \psi^k(S(x_0, x_0, x_1)) < \infty.$$

Thus, we have

$$\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0.$$

This implies that $\{x_n\}$ is a S-Cauchy sequence in the S-metric space (X, S). Since (X, S) is complete, there exists $u \in X$ such that $\{x_n\}$ is S-convergent to u. Since T is S-continuous, it follows that $\{Tx_n\}$ is S-convergent to Tu. By the uniqueness of the limit, we get $u = Tu$, that is u is a fixed point of T. □

Definition 2.3. Let (X, S) be a S-metric space and $T : X \to X$ be a given mapping. We say that T is a $S - \beta - \psi$ contractive mapping of type I if there exist two functions $\beta : X \times X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y, z \in X$, we have

$$\beta(x, y, z)S(Tx, Ty, Tz) \leq \psi(S(x, y, z)).$$

Corollary 2.1. *Let (X, S) be a complete S-metric space. Suppose that $T : X \to X$ is a $S - \beta - \psi$ contractive mapping of type I and satisfies the following conditions.*

(i) *T is β-admissible;*
(ii) *there exists $x_0 \in X$ such that $\beta(x_0, x_0, Tx_0) \geq 1$;*
(iii) *T is S-continuous.*

Then there exists $u \in X$ such that $Tu = u$.

Example 2.1. Let $X = [0, \infty)$ be endowed with the S-metric

$$S(x, y, z) = |x + y - 2z| \text{ for all } x, y \in X.$$

Define $T : X \to X$ by $Tx = 3x$ for all $x \in X$. We define $\beta : X \times X \times X \to [0, \infty)$ in the following way. $\beta(x, y, z) = \begin{cases} \frac{1}{9}, & \text{if } (x, y, z) \notin (0, 0, 0); \\ 1, & \text{otherwise.} \end{cases}$
One can easily show that

$$\beta(x, y, z)S(Tx, Ty, Tz) \leq \frac{1}{9} S(x, y, z) \text{ for all } x, y \in X.$$

Then T is a $S - \beta - \psi$ contractive mapping of the type I with $\psi(t) = \frac{1}{9}t$ for all $t \in [0, \infty)$. Take $x, y, z \in X$ such that $\beta(x, y, z) \geq 1$. By the definition of T, this implies that $x = y = z = 0$. Then we have $\beta(Tx, Ty, Tz) = \beta(0, 0, 0) = 1$, and so T is β-admissible. All the condition of Corollary (2.1) are satisfied. Here, 0 is the fixed point of T.

The following results can be easily concluded from Theorem (2.1).

Corollary 2.2. *Let (X, S) be a complete S-metric space. Suppose that $T : X \to X$ is a generalised $S-\beta-\psi$ contractive mapping of type II and satisfies the following conditions.*

(i) *T is β-admissible;*
(ii) *there exists $x_0 \in X$ such that $\beta(x_0, x_0, Tx_0) \geq 1$;*
(iii) *T is S-continuous.*

Then there exists $u \in X$ such that $Tu = u$.

The next theorem does not require the continuity of T.

Theorem 2.2. *Let (X, S) be a complete S-metric space. Suppose that $T : X \to X$ is a generalised $S - \beta - \psi$ contractive mapping of type I such that ψ is continuous and satisfies the following conditions.*

(i) *T is β-admissible;*
(ii) *there exists $x_0 \in X$ such that $\beta(x_0, x_0, Tx_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\}$ is a S-convergent to $x \in X$, then $\beta(x_n, x_n, x) \geq 1$ for all n.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem (2.1), we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, is a S-Cauchy sequence in the complete S-metric space (X, S) that is S-convergent to $u \in X$. From (2) and (iii) we have

$$\beta(x_n, x_n, u) \geq 1, \tag{4}$$

for all $n \geq 0$.

Using (4), we have

$$\begin{aligned} S(x_{n+1}, x_{n+1}, Tu) &= S(Tx_n, Tx_n, Tu) \\ &\leq \beta(x_n, x_n, u) S(Tx_n, Tx_n, Tu) \\ &\leq \psi(M(x_n, x_n, u)), \end{aligned}$$

where

$$\begin{aligned} M(x_n, x_n, u) &= \max\{S(x_n, x_n, u), S(x_n, x_n, Tx_n), S(u, u, Tu), \\ &\quad \frac{1}{4}(S(x_n, x_n, Tx_n) + S(x_n, x_n, Tu) + S(u, u, Tx_n)) \\ &= \max\{S(x_n, x_n, u), S(x_n, x_n, x_{n+1}), S(u, u, Tu)\} \\ &\quad \frac{1}{4}(S(x_n, x_n, x_{n+1}) + S(x_n, x_n, Tu) + S(u, u, x_{n+1})). \end{aligned}$$

Letting $n \to \infty$ in the following inequality

$$S(x_{n+1}, x_{n+1}, Tu) \leq \psi(M(x_n, x_n, u)).$$

It follows that

$$S(u, u, Tu) \leq \psi(S(u, u, Tu)),$$

which is a contradiction. Thus, we obtain $S(u, u, Tu) = 0$ and hence $u = Tu$. □

The following corollary can be easily derived from Theorem (2.2).

Corollary 2.3. *Let (X,S) be a complete S-metric space. Suppose that $T : X \to X$ is a generalised $S-\beta-\psi$ contractive mapping of type II such that ψ is continuous and satisfies the following condition.*

(i) *T is β-admissible;*
(ii) *there exists $x_0 \in X$ such that $\beta(x_0, x_0, Tx_0) \geq 1$;*
(iii) *If $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\}$ is a S-convergent to $x \in X$, then $\beta(x_n, x_n, x) \geq 1$ for all n.*

Then there exists $u \in X$ such that $Tu = u$.

Theorem 2.3. *Adding the following condition to the hypotheses of Theorem* (2.1) *(resp. Theorem* (2.2)*, Corollary* (2.2)*, Corollary* (2.3)*), we obtain the uniqueness of the fixed point of T.*

(iv) *For $x \in Fix(T)$, $\beta(x, x, z) \geq 1$ for all $z \in X$.*

Proof. Let $u, v \in Fix(T)$ be two fixed points of T. By (iv), we derive $\beta(u, u, v) \geq 1$. Notice that $\beta(Tu, Tu, Tv) = \beta(u, u, v)$ since u and v are fixed points of T. Consequently, we have

$$\begin{aligned} S(u,u,v) &= S(Tu,Tu,Tv) \\ &\leq \beta(u,u,v)S(Tu,Tu,Tv) \leq \psi(M(u,u,v)), \end{aligned}$$

where

$$\begin{aligned} M(u,u,v) &= \max\{S(u,u,v), S(u,u,Tu), S(v,v,Tv), \\ &\quad \frac{1}{4}(S(u,u,Tu) + S(u,u,Tv) + S(v,v,Tu))\} \\ &= \max\{S(u,u,v), \frac{1}{4}(S(u,u,v) + S(v,v,u))\} \\ &= S(u,u,v). \end{aligned}$$

Thus, we get that

$$\begin{aligned} S(u,u,v) &\leq \psi(M(u,u,v)) \\ &\leq \psi(S(u,u,v)) \\ &< S(u,u,v), \end{aligned}$$

which is a contradiction. Therefore, $u = v$, that is, the fixed point of T is unique. □

Corollary 2.4. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a given mapping. Suppose that there exists a continuous function $\psi \in \Psi$ such that*

$$S(Tx, Ty, Tz) \leq \psi(M(x, y, z)),$$

for all $x, y, z \in X$. Then T has a unique fixed point.

Corollary 2.5. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a given mapping. Suppose that there exists a continuous function $\psi \in \Psi$ such that*

$$S(Tx,Ty,Tz) \leq \psi(S(x,y,z)),$$

for all $x,y,z \in X$. Then T has a unique fixed point.

Corollary 2.6. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a given mapping. Suppose that there exists $\lambda \in [0,1)$ such that*

$$\begin{aligned} S(Tx,Ty,Tz) \leq\ & \lambda \max\{S(x,y,z), S(x,x,Tx), S(y,y,Ty), \\ & S(z,z,Tz), \frac{1}{4}(S(x,x,Ty)+S(y,y,Tz)+S(z,z,Tx))\}, \end{aligned}$$

for all $x,y,z \in X$. Then T has a unique fixed point.

Corollary 2.7. *Let (X,S) be a complete S-metric space and let $T : X \to X$ be a given mapping. Suppose that there exist nonnegative real numbers a,b,c,d,e with $a+b+c+d+e<1$ such that*

$$\begin{aligned} S(Tx,Ty,Tz) \leq\ & aS(x,y,z)+bS(x,x,Tx) \\ & +cS(y,y,Ty)+d(S(z,z,Tz)) \\ & +\frac{e}{4}(S(x,x,Ty)+S(y,y,Tz)+S(z,z,Tx)), \end{aligned}$$

for all $x,y,z \in X$. Then T has a unique fixed point.

Corollary 2.8. *Let (X,S) be a complete S-metric space and let $T : X \to X$ be a given mapping. Suppose that there exists $\lambda \in [0,1)$ such that*

$$S(Tx,Ty,Tz) \leq \lambda S(x,y,z),$$

for all $x,y,z \in X$. Then T has a unique fixed point.

3. Consequences

Fixed point theorems on metric spaces endowed with a partial order.

Definition 3.1. Let $(X,\preceq)$ be a partially ordered set and $T : X \to X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if

$$x,y \in X,\ x \preceq y \Rightarrow Tx \preceq Ty.$$

Definition 3.2. Let $(X,\preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.

Definition 3.3. Let $(X,\preceq)$ be a partially ordered set and S be a S-metric space on X. We say $(X,\preceq,S)$ a S-regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, $x_n \preceq x$ for all n.

Theorem 3.1. *Let $(X,\preceq)$ be a partially ordered set and $\preceq$ be a S-metric on X such that (X,S) is a complete S-metric space. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$S(Tx,Tx,Ty) \preceq \psi(M(x,x,y)), \tag{5}$$

for all $x,y \in X$ with $x \preceq y$. Suppose also that the following conditions hold.

(i) *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is S-continuous or $(X,\preceq,S)$ is S-regular and ψ is continuous.*

Then there exists $u \in X$ such that $Tu = u$. Moreover, if for $x \in Fix(T)$, $x \preceq z$ for all $z \in X$, one has the uniqueness of the uniqueness of the fixed point.

Proof. Define the mapping $\beta : X \times X \times X \to [0,\infty)$ by

$$\beta(x,x,y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

From (5), for all $x,y \in X$, we have

$$\beta(x,x,y)S(Tx,Tx,Ty) \leq \psi(M(x,x,y)).$$

It follows that T is a generalised $S-\beta-\psi$ contractive mapping of type II. From the condition (ii), we have

$$\beta(x_0,x_0,Tx_0) \geq 1.$$

By the definition of β and since T is a nondecreasing mapping with respect to $\preceq$, we have

$$\beta(x,x,y) \geq 1 \Rightarrow x \preceq y \Rightarrow Tx \preceq Ty \Rightarrow \beta(Tx,Tx,Ty) \geq 1.$$

Thus T is β-admissible. Moreover, if T is S-continuous, by Theorem (2.1), T has a fixed point.

Now, suppose that $(X,\preceq,S)$ is S-regular. Let $\{x_n\}$ be a sequence in X such that $\beta(x_n,x_n,x_{n+1}) \geq 1$ for all n and x_n is S-convergent to $x \in X$. By Definition (3.3), $x_n \preceq x$ for all n, which gives us $\beta(x_n,x_n,x) \geq 1$ for all k. Thus, all the hypothesis of Theorem (2.2) are satisfied and there exists $u \in X$ such that $Tu = u$. To prove the uniqueness, since $u \in Fix(T)$, we have, $u \preceq z$ for all $z \in X$. By the definition of β, we get that $\beta(u,u,z) \geq 1$ for all $z \in X$. Therefore, the hypothesis (iv) of Theorem (2.3) is satisfied and deduce the uniqueness of the fixed point.

□

Corollary 3.1. *Let $(X,\preceq)$ be a partially ordered set and S be a S-metric on X such that (X,S) is a complete S-metric space. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$S(Tx,Tx,Ty) \leq \psi(S(x,x,y)),$$

for all $x,y \in X$ with $x \preceq y$. Suppose also that the following conditions hold.

(i) *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is S-continuous or $(X, \preceq, S)$ is S-regular.*

Then there exists $u \in X$ such that $Tu = u$. Moreover, if for $x \in Fix(T)$, $x \preceq z$ for all $z \in X$, one has the uniqueness of the fixed point.

Corollary 3.2. *Let $(x, \preceq)$ be a partially ordered set and S be a S-metric space on X such that (X, S) is a complete S-metric space. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist nonnegative real numbers a, b, c, and d with $a + b + c + d < 1$ such that*

$$\begin{aligned} S(Tx, Tx, Ty) \leq \; & aS(x, x, y) + bS(x, x, Tx) + cS(y, y, Ty) \\ & + \frac{d}{4}(S(x, x, Tx) + S(x, x, Ty) + S(y, y, Tx)), \end{aligned}$$

for all $x, y \in X$ with $x \preceq y$. Suppose also that the following conditions hold.

(i) *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is S-continuous or $(X, \preceq, S)$ is S-regular.*

Then there exists $u \in X$ such that $Tu = u$. Moreover, if for $x \in Fix(T)$, $x \preceq z$ for all $z \in X$, one has the uniqueness of the fixed point.

Corollary 3.3. *Let $(x, \preceq)$ be a partially ordered set and S be a S-metric space on X such that (X, S) is a complete S-metric space. Let $T : X \to X$ is a nondecreasing mapping with respect to $\preceq$. Suppose that there exist a constant $\lambda \in [0, 1)$ such that*

$$S(Tx, Tx, Ty) \leq \lambda S(x, x, y),$$

for all $x, y \in X$ with $x \preceq y$. Suppose also that the following conditions hold.

(i) *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is S-continuous or $(X, \preceq, S)$ is S-regular.*

Then there exists $u \in X$ such that $Tu = u$. Moreover, if for $x \in Fix(T)$, $x \preceq z$ for all $z \in X$, one has the uniqueness of the fixed point.

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References

[1] V. Berinde, *Iterative Approximation of Fixed Points*, Editura Efemeride, Baia Mare (2002).

[2] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha - \psi-$contractive type mappings*, Nonlinear Analysis: Theory, Methods and Applications, 75 (4) (2012), 2154–2165.

[3] E. Karapinar, B. Samet, *Generalized $(\alpha - \psi)$ contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal., 2012 (2012), Article ID: 793486, 17 pages.

[4] M. A. Alghamdi, E. Karapinar, *$G-\beta-\psi$ contractive-type mappings and related fixed point theorems*, Journal of Inequalities and Applications, 2013, 2013:70.

[5] M. Zhou, X. Liu, S. Radenović, *$S-\gamma-\phi-\psi$-contractive type mappings in S-metric spaces*, J. Nonlinear Sci. Appl., 10 (2017), 1613–1639.

[6] N. Priyobarta, Y. Rohen, N. Mlaiki, *Complex valued S_b-metric spaces*, Journal of Mathematical Analysis, 8 (3) (2017), 13–24.

[7] N. Y. Özgür, N. Taş, *Some fixed point theorems on S-metric spaces*, Mat. Vesnik, 69 (1) (2017), 39–52.

[8] N. Y. Özgür, N. Taş, *New contractive conditions of integral type on complete S-metric spaces*, Math Sci, (2017) 11:231-240.

[9] S. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik, 64 (2012), 258–266.

[10] Y. Rohen, T. Došenović, S. Radenović, *A Note on the Paper "A Fixed Point Theorems in S_b-Metric Spaces"*, Filomat, 31 (11) (2017), 3335–3346.

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