Mathematica Moravica
Vol. **22**, No. **1** (2018), 107–121 doi: 10.5937/MatMor1801107M

(f,g)-derivation of ordered Γ-semirings

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Abstract. In this paper, we introduce the concept of (f,g)-derivation, which is a generalization of $f-$ derivation and derivation of ordered Γ-semiring and study some properties of $(f,g)-$derivation of ordered Γ-semirings. We prove that, if d is a (f,g)-derivation of an ordered integral Γ-semiring M then $\ker d$ is a $m-k-$ideal of M and we characterize $m-k-$ideal using (f,g)-derivation of ordered Γ-semiring M.

1. Introduction

In 1995, Murali Krishna Rao [14, 15, 16] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring. Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages etc. The notion of a semiring was first introduced by Vandiver [23] in 1934 but semirings had appeared in studies on the theory of ideals of rings. A universal algebra $S=(S,+,\cdot)$ is called a semiring if and only if $(S,+),(S,\cdot)$ are semigroups which are connected by distributive laws, $a(b+c)=ab+ac,\ (a+b)c=ac+bc,$ for all $a,b,c\in S$. A natural example of semiring is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if I is the unit interval on the real line, then $(I,\max,\min)$ is a semiring in which 0 is the additive identity and 1 is the mutiplicative identity. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. Many semirings have order structure in addition to their algebraic structure. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring. Semirings are useful in the areas of theoretical computer science as well as in the solutions of

2010 *Mathematics Subject Classification.* Primary: 16Y60, 06F35, 08A30, 03G25.

Key words and phrases. Ordered Γ-semiring, derivation, Γ-semiring, integral ordered Γ-semiring, (f,g)-derivation.

Full paper. Received 29 December 2017, revised 28 February 2018, accepted 1 June 2018, available online 25 June 2018.

graph theory and optimization theory in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches. For an overview on semirings and their applications see [2, 3, 4, 5, 6, 7, 8, 11].

As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [21] in 1964. In 1981 Sen [20] introduced the notion of $\Gamma-$semigroup as a generalization of semigroup. The notion of ternary algebraic system was introduced by Lehmer [12] in 1932. Lister [13] introduced ternary ring. The set of all negative integers Z is not a semiring with respect to usual addition and multiplication but Z forms a $\Gamma-$semiring where $\Gamma = Z$. The important reason for the development of $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, semigroups and ternary semirings. Murali Krishna Rao and Venkateswarlu [17] introduced the notion of $\Gamma-$incline and field $\Gamma-$semiring and studied properties of regular $\Gamma-$incline and field $\Gamma-$semiring.

Over the last few decades, several authors have investigated the relationship between the commutativity of ring R and the existence of certain specified derivations of R. The first result in this derivation is due to Posner [22] in 1957. In the year 1990, Bresar and Vukman [1] established that a prime ring must be commutative if it admits a nonzero left derivation. Kim [9, 10] studied right derivation and generalized derivation of incline algebra. The notion of derivation of algebraic structures is useful for characterization of algebraic structures. The notion of derivation has also been generalized in various directions such as right derivation, left derivation, $f-$derivation, reverse derivation, orthogonal derivation, (f, g)-derivation, generalized right derivation, etc. Murali Krishna Rao and Venkateswarlu [18, 19] introduced the notion of generalized right derivation of $\Gamma-$ incline and right derivation of ordered $\Gamma-$semiring. A ring R is said to be $n-$torsion free if for any $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ and d is said to be a Jordan derivation if $d(x^2) = d(x)x + xd(x)$, for all $x, y \in R$. Every derivation is a Jordan derivation but the converse need not be true. In this paper, we introduce the concept of (f, g)-derivation which is a generalization of $f-$derivation and derivation of ordered $\Gamma-$semirings and study some properties of (f, g)-derivation of ordered Γ-semirings.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions necessary for this paper.

Definition 2.1. A semiring $(M, +, \cdot)$ is an algebraic structure with two binary operations "$+$" and "$\cdot$" such that $(M, +)$ and $(M, \cdot)$ are semigroups and the distributive laws hold, i.e.,
$x(y + z) = xy + xz$ and $(x + y)z = xz + yz$, for all $x, y, z \in M$.

A semiring $(M, +, \cdot)$ is said to be division semiring if $(M \setminus 0, \cdot)$ is a group.

Definition 2.2. Let M and Γ be two non-empty sets. Then M is called a $\Gamma-$semigroup if it satisfies

(i) $x\alpha y \in M$;
(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.3. Let $(M, +)$ and $(\Gamma, +)$ be semigroups. A $\Gamma-$semigroup M is said to be $\Gamma-$semiring M if it satisfies the following axioms, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$.

Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation as the usual semiring multiplication.

Definition 2.4. A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0+x = x = x+0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M$.

Definition 2.5. A $\Gamma-$semiring M with zero element 0 is said to be hold cancellation laws if $a+b = a+c, b+a = c+a$, where $a, b, c \in M$, then $b = c$.

Definition 2.6. An element $a \in M$ is said to be an idempotent of $\Gamma-$semigroup M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be $\alpha-$idempotent.

Definition 2.7. A semigroup $(M, +)$ is said to be a band if $a + a = a$, for all $a \in M$.

Definition 2.8. An element $a \in M$ is said to be an idempotent of $\Gamma-$semiring M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α-idempotent.

Definition 2.9. If every element of a $\Gamma-$semiring M is an idempotent of M and if semigroup $(M, +)$ is a band then M is said to be idempotent $\Gamma-$semiring M.

Definition 2.10. A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M, \alpha \in \Gamma$.

Let M be a $\Gamma-$semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.11. A $\Gamma-$semiring M is said to be commutative $\Gamma-$semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.12. A non zero element a in a $\Gamma-$semiring M with zero is said to be a zero divisor if there exits non zero element $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 0$.

A $\Gamma-$semiring M with unity 1 and zero element 0 is called integral $\Gamma-$semiring if it has no zero divisors.

Definition 2.13. A $\Gamma-$semiring M is called an ordered $\Gamma-$semiring if it admits a compatible relation $\leq$, i.e., $\leq$ is a partial ordering on M which satisfies the following conditions. If $a \leq b$ and $c \leq d$ then, for all $a, b, c, d \in M, \alpha \in \Gamma$, hold

(i) $a + c \leq b + d, c + a \leq d + b$,
(ii) $a\alpha c \leq b\alpha d$,
(iii) $c\alpha a \leq d\alpha b$.

Example 2.1. Let $M = [0, 1], \Gamma = N$, $+$ and ternary operation be defined as $x + y = \max\{x, y\}, x\gamma y = \min\{x, \gamma, y\}$ for all $x, y \in M, \gamma \in \Gamma$. Then M is an ordered $\Gamma-$semiring with respect to usual ordering.

Example 2.2. Let M be the additive semigroup of all 2×2 matrices, where elements belong to $N \cup \{0\}$, Γ be the additive semigroup of all 2×2 matrices, whose elements belong to N and ternary operation is defined as $M \times \Gamma \times M \to M$ by $(x, \alpha, y) \to x\alpha y$ using usual matrix multiplication for all $x, y \in M$ and $\alpha \in \Gamma$. Let $A = (a_{ij})^2_{i,j=1}, B = (b_{ij})^2_{i,j=1} \in M$, we define $A \leq B$ if and only if $a_{ij} \leq b_{ij}$, for all $i, j \in \{1, 2\}$. Then M is an ordered $\Gamma-$semiring.

Definition 2.14. An ordered $\Gamma-$semiring M is said to be totally ordered $\Gamma-$semiring M if any two elements of M are comparable.

Definition 2.15. In an ordered $\Gamma-$semiring M:

(i) the semigroup $(M, +)$ is said to be positively ordered , if $a \leq a + b$ and $b \leq a + b$, for all $a, b \in M$,
(ii) the semigroup $(M, +)$ is said to be negatively ordered, if $a + b \leq a$ and $a + b \leq b$, for all $a, b \in M$,
(iii) the $\Gamma-$semigroup M is said to be positively ordered, if $a \leq a\alpha b$ and $b \leq a\alpha b$, for all $\alpha \in \Gamma, a, b \in M$,
(iv) $\Gamma-$semigroup M is said to be negatively ordered if $a\alpha b \leq a$ and $a\alpha b \leq b$ for all $\alpha \in \Gamma, a, b \in M$.

Definition 2.16. A non-empty subset A of an ordered $\Gamma-$semiring M is called a $\Gamma-$subsemiring M if $(A, +)$ is a subsemigroup of $(M, +)$ and $a\alpha b \in A$, for all $a, b \in A$ and $\alpha \in \Gamma$.

Definition 2.17. Let M be an ordered $\Gamma-$semiring. A non-empty subset I of M is called a left (right) ideal of ordered $\Gamma-$semiring M if I is closed under addition, $M\Gamma I \subseteq I$ $(I\Gamma M \subseteq I)$ and if for any $a \in M$, $b \in I$, $a \leq b$ implies $a \in I$.

I is called an ideal of M if it is both a left ideal and a right ideal of M.

A non-empty subset A of ordered $\Gamma-$semiring M is called a $k-$ideal if A is an ideal and $x \in M$, $x + y \in A, y \in A$ implies $x \in A$.

Definition 2.18. Let M and N be ordered $\Gamma-$ semirings. A mapping $f : M \to N$ is called a homomorphism if:

(i) $f(a+b) = f(a) + f(b)$,
(ii) $f(a\alpha b) = f(a)\alpha f(b)$,

for all $a, b \in M, \alpha \in \Gamma$.

Definition 2.19. Let M be an ordered $\Gamma-$semiring. A mapping $f : M \to M$ is an isotone mapping of M if $x \leq y$ implies $f(x) \leq f(y)$, for all $x, y \in M$.

Definition 2.20. Let M be an ordered Γ-semiring with zero element 0. M is said to be multiplicatively cancellative if $a \neq 0, a\alpha b = a\alpha c, b\alpha a = c\alpha a$, where $a, b, c \in M, \alpha \in \Gamma$ implies $b = c$.

Let M be an ordered Γ-semiring. M is said to be is said to be additively cancellative if $a + b = a + c, b + a = c + a$, where $a, b, c \in M$, implies $b = c$.

Definition 2.21. Let M be an ordered $\Gamma-$ semiring. A mapping $f : M \to M$ is called an endomorphism if:

(i) f is an onto,
(ii) $f(a+b) = f(a) + f(b)$,
(iii) $f(a\alpha b) = f(a)\alpha f(b)$,

for all $a, b \in M$ and $\alpha \in \Gamma$.

3. (f, g)-derivation of ordered $\Gamma-$semirings

In this section we introduce the concept of (f, g)-derivation of ordered Γ-semirings and we study some of their properties.

Definition 3.1. Let M be an ordered Γ-semiring. A mapping $d : M \to M$ is called a derivation if it satisfies:

(i) $d(x+y) = d(x) + d(y)$,
(ii) $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$,

for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 3.2. Let M be an ordered Γ-semiring and f be an endomorphism on M. A mapping $d : M \to M$ is called an f-derivation if it satisfies:

(i) $d(x+y) = d(x) + d(y)$,
(ii) $d(x\alpha y) = d(x)\alpha f(y) + f(x)\alpha d(y)$,

for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 3.3. Let M be an ordered Γ-semiring and f, g be two endomorphisms on M. A mapping $d : M \to M$ is called an (f, g)-derivation if it satisfies:

(i) $d(x+y) = d(x) + d(y)$,
(ii) $d(x\alpha y) = d(x)\alpha f(y) + g(x)\alpha d(y)$,

for all $x, y \in M$ and $\alpha \in \Gamma$.

Example 3.1. $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in N \cup \{0\} \right\}$ be an additive semigroup and $\Gamma = \left\{ \begin{pmatrix} a & a \\ 0 & a \end{pmatrix} \mid a \in N \right\}$ be an additive semigroup and ternary operation defined as usual matrix multiplication.

Let $A = (a_{ij})$, $B = (b_{ij}) \in M$ and let $A \leq B$ if and only if $a_{ij} \leq b_{ij}$, for all $i, j \in \{1, 2\}$.

Define $d : M \to M,\ f : M \to M$ and $g : M \to M$ by

$$d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix};\ f\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix};\ g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

Then d is a $(f, g)-$derivation of an ordered $\Gamma-$semiring M.

Remark 3.1. Let d be a (f, g)-derivation of an ordered Γ-semiring M. If $f(x) = x$ and $g(x) = x$, for all $x \in M$ then d is a derivation of M. Indeed, let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$d(x\alpha y) = d(x)\alpha f(y) + g(x)\alpha d(y) = d(x)\alpha y + x\alpha d(y).$$

Hence d is a derivation of M. Therefore (f, g)-derivation d of M is a generalization of derivation d of M.

Theorem 3.1. *Let d be a (f, g)-derivation of an idempotent commutative ordered Γ-semiring M in which Γ semigroup M is negatively ordered and semigroup $(M, +)$ is positively ordered .*
If $f(x) \leq g(x)$, for all $x \in M$ then $d(x) \leq g(x)$, for all $x \in M$.

Proof. Suppose $f(x) \leq g(x)$, for all $x \in M$. Then $f(x)+g(x) \leq g(x)+g(x)$, and therefore $f(x) + g(x) \leq g(x) \leq f(x) + g(x)$, so, $f(x) + g(x) = g(x)$.

Let $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x = x\alpha x$, so we have

$$\begin{aligned} d(x) &= d(x\alpha x) = d(x)\alpha f(x) + g(x)\alpha d(x) = d(x)\alpha[f(x) + g(x)] \\ &= d(x)\alpha[g(x)] \leq g(x). \end{aligned}$$

Thus, $d(x) \leq g(x)$, for all $x \in M$. □

Theorem 3.2. *Let d be a (f, g)-derivation of an ordered Γ-semiring M. If $f(0) = g(0) = 0$ then $d(0) = 0$.*

Proof. Suppose d is a (f, g)-derivation of M. Then

$$d(0) = d(0\alpha 0) = d(0)\alpha f(0) + g(0)\alpha d(0) = d(0)\alpha 0 + 0\alpha d(0) = 0 + 0 = 0,$$

and therefore holds $d(0) = 0$. □

Theorem 3.3. *Let f, g be two endomorphisms on idempotent commutative ordered Γ-semiring M in which $\Gamma-$ semigroup M is negatively ordered, semigroup $(M, +)$ is positively ordered and $g(x) \leq f(x)$, for all $x \in M$. Then f is a (f, g)-derivation of M.*

Proof. Let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} f(x\alpha y) &= f(x)\alpha f(y) = f(x)\alpha f(y) + f(x)\alpha f(y) \\ &= f(x)\alpha f(y) + [g(x) + f(x)]\alpha f(y) \\ &= f(x)\alpha f(y) + g(x)\alpha f(y) + f(x)\alpha f(y) \\ &= f(x)\alpha f(y) + g(x)\alpha f(y). \end{aligned}$$

Also, we have $f(x+y) = f(x)+f(y)$, since f is endomorphism of M. Hence f is a derivation of M. □

Theorem 3.4. *Let I be a nonzero ideal of an integral ordered Γ-semiring M in which Γ-semigroup M is negatively ordered. If d is a nonzero $(f, g)-$ derivation and an isotone mapping on M, where g is a nonzero function on I then d is a nonzero $(f, g)-$ derivation on I.*

Proof. Let d be a $(f, g)-$ derivation on I and g be a nonzero function on I. Suppose that $x \in I$ such that $g(x) \neq 0$ $d(x) = 0$ and $y \in M$ and $\alpha \in \Gamma$. We have $x\alpha y \leq x$ and $d(x\alpha y) \leq d(x)$, which implies $d(x\alpha y) = 0$, so we have $d(x)\alpha f(y) + g(x)\alpha d(y) = 0$, i.e., $g(x)\alpha d(y) = 0$. Therefore $d(y) = 0$, since M is an integral ordered Γ-semiring. This contradicts that d is a nonzero (f, g)- derivation on M. Hence, d is a nonzero (f, g)-derivation on I. □

Theorem 3.5. *Let M be an idempotent ordered Γ-semiring and d be a (f, g)-derivation on M. If $d \circ d = d$ and $f \circ d = f$, then for each $x \in M$ there exists $\alpha \in \Gamma$ such that $d(x\alpha d(x)) = d(x)$.*

Proof. Let $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x = x\alpha x$, and

$$\begin{aligned} d(x\alpha d(x)) &= d(x)\alpha f(d(x)) + g(x)\alpha d(d(x)) \\ &= d(x)\alpha f(x) + g(x)\alpha d(x) = d(x\alpha x) = d(x). \end{aligned}$$

Thus, $d(x\alpha d(x)) = d(x)$. □

Definition 3.4. An ordered $\Gamma-$semiring M is called a prime ordered $\Gamma-$semiring if $a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$, for all $a, b \in M$.

Definition 3.5. An ordered $\Gamma-$semiring M is called a $2-$torsion free if $2a = 0$ implies $a = 0$, for all $a \in M$.

Theorem 3.6. *Let M be a prime ordered Γ-semiring and I be a nonzero ideal of M. If there exists (f, g)- derivation d on M such that $g(x) = x$, for all $x \in M$, and $d(I)\alpha x = 0$ then $x = 0$.*

Proof. Suppose $d(I)\alpha x = 0$, $\alpha \in \Gamma$. Then, for all $\gamma \in I, a, b \in M$ and $\alpha \in \Gamma$ holds $d(\gamma\alpha a)\alpha x = 0$. From this we have

$$[d(\gamma)\alpha f(a) + g(\gamma)\alpha d(a)]\alpha x = 0,$$
$$d(\gamma)\alpha f(a)\alpha x + g(\gamma)\alpha d(a)\alpha x = 0,$$

$$d(\gamma)\alpha f(a)\alpha x+\gamma\alpha d(a)\alpha x=0,$$
$$0+\gamma\alpha d(a)\alpha x=\gamma\alpha d(a)\alpha x=0.$$

Further, if we replace a by $\gamma\alpha b$, we will obtain

$$\gamma\alpha d(\gamma\alpha b)\alpha x=\gamma\alpha[d(\gamma)\alpha f(b)+g(\gamma)\alpha d(b)]\alpha x=0,$$
$$\gamma\alpha\gamma\alpha d(b)\alpha x=d(b)\alpha x=0,$$

and since $d\neq 0$, we have $x=0$. □

Theorem 3.7. *Let M be a $2-$torsion free prime ordered Γ-semiring, d be an (f,g)- derivation on M such that $f\circ d=d\circ f$ and $g(x)=x$, for all $x\in M$. If $d^2=0$ then $d=0$.*

Proof. Suppose $d^2=0$, $x,y\in M$ and $\alpha\in\Gamma$. Then $d^2(x\alpha y)=0$, and the following holds

$$d[d(x)\alpha f(y)+g(x)\alpha d(y)]=0,$$
$$d^2(x)\alpha f(f(y))+g(d(x))\alpha d(f(y))d(g(x))\alpha f(d(y))+g(g(x))\alpha d(d(y))=0,$$
$$d(x)\alpha d(f(y))+d(x)\alpha f(d(y))=d(x)\alpha[d(f(y))+f(d(y))]=0,$$
$$d(x)\alpha[2d(f(y))]=0.$$

Therefore $d(x)=0$, for all $x\in M$. Hence $d=0$. □

Theorem 3.8. *Let d be a (f,g)- derivation on a prime ordered Γ-semiring M and $g(x)=x$, for all $x\in M$. If $a\in M$, $\alpha\in\Gamma$ such that $a\alpha d(x)=0$ or $d(x)\alpha a=0$ then $a=0$ or $d=0$.*

Proof. Let $x,y\in M$, $\alpha\in\Gamma$. Suppose that $a\alpha d(x)=0$, for all $x\in M$. Then $a\alpha d(x\alpha y)=0$. So,

$$a\alpha[d(x)\alpha f(y)+g(x)\alpha d(y)]=0,$$

and therefore

$$a\alpha d(x)\alpha f(y)+a\alpha g(x)\alpha d(y)=a\alpha d(x)\alpha f(y)+a\alpha x\alpha d(y)]=0,$$

so we obtain $a\alpha x\alpha d(y)=0$. Hence $a=0$ or $d=0$.

Similarly, $d(x)\alpha a=0$ implies $a=0$ or $d=0$. □

Theorem 3.9. *Let d be a (f,g)- derivation of an ordered idempotent Γ-semiring M. If $d\circ d=d$ and $f\circ d=f$ then for each $x\in M$ there exist $\alpha\in\Gamma$ such that $d(x\alpha d(x))=d(x)$.*

Proof. Suppose d is a (f,g)- derivation of ordered idempotent Γ-semiring M such that $d\circ d=d$, $f\circ d=f$ and $x\in M$. Then there exists $\alpha\in\Gamma$ such that $x\alpha x=x$. Now,

$$\begin{aligned} d(x\alpha d(x))&=d(x)\alpha f(d(x))+g(x)\alpha d(d(x))\\ &=d(x)\alpha f(x)+g(x)\alpha d(x)=d(x\alpha x)=d(x). \end{aligned}$$

Therefore holds $d(x\alpha d(x))=d(x)$. □

Theorem 3.10. *Let M be an ordered Γ-semiring in which $(M,+)$ is cancellative. Let d be a (f,g)- derivation of M, $g(x)=x$, for all $x \in M$, let$I$ be a Γ-subsemiring of M such that $f(I)=I$ and $d(x\alpha y)=d(x)\alpha d(y)$, for all $x,y \in I$, $\alpha \in \Gamma$. Then $d(x)\alpha y\alpha f(x)=x\alpha y\alpha d(x)=d(x)\alpha y\alpha d(x)$, for all $x,y \in I$.*

Proof. Let $x,y \in I$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} d(x\alpha y\alpha x) &= d(x)\alpha f(y\alpha x)+x\alpha d(y\alpha x) \\ &= d(x)\alpha f(y)\alpha f(x)+x\alpha d(y)\alpha d(x), \end{aligned} \tag{1}$$

$$\begin{aligned} d(x\alpha y\alpha x) &= [d(x)\alpha f(y)+x\alpha d(y)]\alpha d(x) \\ &= d(x)\alpha f(y)\alpha d(x)+x\alpha d(y)\alpha d(x). \end{aligned} \tag{2}$$

From (1) and (2), we have $d(x)\alpha f(y)\alpha f(x)=d(x)\alpha f(y)\alpha d(x)$. Hence $d(x)\alpha z\alpha f(x)=d(x)\alpha z\alpha d(x)$, for all $x,z \in I$.

Now,

$$\begin{aligned} d(y\alpha x\alpha y) &= d(y\alpha x)\alpha f(y)+g(y\alpha x)\alpha d(y) \\ &= d(y)\alpha d(x)\alpha f(y)+y\alpha x\alpha d(y), \end{aligned} \tag{3}$$

$$\begin{aligned} d(y\alpha x\alpha y) &= d(y)\alpha d(x\alpha y) \\ &= d(y)\alpha[d(x)\alpha f(y)+g(x)\alpha d(y)] \\ &= d(y)\alpha d(x)\alpha f(y)+d(y)\alpha g(x)\alpha d(y) \\ &= d(y)\alpha d(x)\alpha f(y)+d(y)\alpha x\alpha d(y). \end{aligned} \tag{4}$$

From (3) and (4), we get $d(y)\alpha x\alpha d(y)=y\alpha x\alpha d(y)$, for all $y \in I$, and therefore $d(x)\alpha y\alpha d(x)=x\alpha y\alpha d(x)$, for all $y \in I, \alpha \in \Gamma$.

Hence

$$d(x)\alpha y\alpha d(x)=x\alpha y\alpha d(x)=d(x)\alpha y\alpha f(x),$$

for all $x,y \in I$. □

Theorem 3.11. *Let M be a commutative ordered Γ-semiring and d_1,d_2 be (f,g)-derivations of M, where $g\circ d_2=g\circ d_1$, $d_1\circ g=d_2\circ g$, $f\circ d_2=f\circ d_1$ $d_1\circ f=d_2\circ f$, $f\circ f=f$ and $g\circ g=g$. Define $d_1d_2(x)=d_1(d_2(x))$, for all $x \in M$. If $d_1d_2=0$ then d_2d_1 is a (f,g)-derivation of M.*

Proof. Suppose $d_1d_2=0$,$x,y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} &d_1d_2(x\alpha y)=d_1[d_2(x)\alpha f(y)+g(x)\alpha d_2(y)]=0, \\ &d_1(d_2(x)\alpha f(y))+d_1(g(x)\alpha d_2(y))=0, \\ &d_1d_2(x)\alpha f(f(y))+g(d_2(x))\alpha d_1(f(y)) \\ &\ +(d_1(g(x)))\alpha f(d_2(y))+g(g(x))\alpha d_1d_2(y)=0. \end{aligned}$$

Therefore

$$g(d_2(x))\alpha d_1(f(y)+d_1(g(x))\alpha f(d_2(y))=0,$$

and hence

$$g(d_1(x))\alpha d_2(f(y)) + d_2(g(x))\alpha f(d_1(y)) = 0.$$

Now, we have

$$\begin{aligned} d_2d_1(x\alpha y) &= d_2[d_1(x\alpha y)] = d_2[d_1(x)\alpha f(y) + g(x)\alpha d_1(y)] \\ &= d_2[d_1(x)\alpha f(y)] + d_2[g(x)\alpha d_1(y)] \\ &= d_2d_1(x)\alpha f(f(y)) + g(d_1(x))\alpha d_2(f(y)) \\ &+ d_2(g(x))\alpha f(d_1(y)) + g(g(x))\alpha d_2(d_1(y)) \\ &= d_2d_1(x)\alpha f(f(y)) + g(g(x))\alpha d_2d_1(y), \\ &= d_2d_1(x)\alpha f(y) + g(x)\alpha d_2d_1(y). \end{aligned}$$

Hence, d_2d_1 is a (f, g)-derivation of M. □

Theorem 3.12. *Let M be an ordered Γ-semiring with unity in which semigroup $(M, +)$ is positively ordered and Γ-semigroup M is negatively ordered and d be a (f, g)-derivation such that $g(x) = x$, for all $x \in M$. If $d(1) = 1$ then $x \leq d(x)$ for all $x \in M$.*

Proof. Let $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x\alpha 1 = x$. Therefore

$$\begin{aligned} d(x) &= d(x\alpha 1) = d(x)\alpha f(1) + g(x)\alpha d(1) \\ &\geq g(x)\alpha d(1) = x\alpha d(1). \end{aligned}$$

Suppose $d(1) = 1$. Then $x\alpha 1 \leq d(x)$, and hence $x \leq d(x)$. □

Theorem 3.13. *Let M be an idempotent ordered Γ-semiring in which semigroup $(M, +)$ is positively ordered and Γ-semigroup M is negatively ordered and d be a (f, g)-derivation such that $f(x) \leq x$ and $g(x) \leq x$, for all $x \in M$. Then $d(x) \leq x$.*

Proof. Let $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x = x\alpha x$, and the following holds

$$\begin{aligned} d(x) &= d(x\alpha x) = d(x)\alpha f(x) + g(x)\alpha d(x) \leq f(x) + g(x) \\ &\leq x + x = x. \end{aligned}$$

Therefore $d(x) \leq x$. □

Theorem 3.14. *Let d be a (f, g)-derivation of idempotent ordered Γ-semiring M. If $d(1) = 1, g(x) \leq x$. Then the following hold for all $x, y \in M$, $\alpha \in \Gamma$:*

(i) $d(x\alpha y) \leq d(x)$,
(ii) $d(x\alpha y) \leq d(y)$,
(iii) *d is an isotone derivation.*

Proof. Let $x, y \in M$ and $\alpha \in \Gamma$.

(i) If $d(x\alpha y) \leq d(x)$, then the following holds.

$$\begin{aligned} d(x\alpha y) &= d(x)\alpha f(y) + g(x)\alpha d(y) \leq d(x)\alpha f(y) + x\alpha d(y) \leq d(x) + x \\ &\leq d(x) + d(x) = d(x). \end{aligned}$$

(ii) Proof of (ii) is similar to (i).

(iii) Let $x \leq y$. Then $x + y = y$ implies $d(x) + d(y) = d(y)$, and therefore $d(x) \leq d(y)$. □

Theorem 3.15. *Let M be an idempotent ordered Γ-semiring with unity in which semigroup $(M, +)$ is positively ordered and Γ-semigroup M is negatively ordered and d be a (f, g)-derivation of M such that $f(x) \leq x$ and $g(x) = x$, for all $x \in M$. Then $d(1) = 1$ if and only if $d(x) = x$.*

Proof. Suppose $d(1) = 1$. By Theorem 3.12, we have $x \leq d(x)$ and by Theorem 3.13, we have $d(x) \leq x$. Therefore $d(x) = x$.

The converse statement can be easily obtained. □

Corollary 3.1. *Let M be an idempotent ordered Γ-semiring in which semigroup $(M, +)$ is positively ordered and Γ-semigroup M is negatively ordered and $d(1) = 1$. Then d is a (f, g)-derivation such that $f(x) \leq x$ and $g(x) = x$, for $x \in M$ if and only if $d(x) = x$.*

Proof. Suppose d is a $(f, g)-$ derivation of ordered $\Gamma-$semiring M such that $f(x) \leq x$ and $g(x) \leq x$, for all $x \in M$. By Theorem 3.15, we have $d(x) = x$.

Conversely, suppose that $d(x) = x$, for $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x\alpha x = x$.

Now, $d(x) = d(x\alpha x)$ impies $x = x\alpha f(x) + g(x)\alpha x$. So, $x \leq f(x) + x \leq f(x)$.

On the other hand, we have $x \geq x\alpha f(x)$ and then $x\alpha x \geq x\alpha f(x)$, and therefore $x \geq f(x)$.

Hence $f(x) = x$. Similarly we can prove $g(x) = x$. □

Theorem 3.16. *Let d be a (f, g)- derivation of an ordered Γ-semiring M with zero and $x \leq y$ implies $x + y = y$, for all $x, y \in M$. Then* $\ker d$ *is a $k-$ideal of M.*

Proof. Let $x, y \in \ker d$ and $\alpha \in \Gamma$. Then $d(x) = d(y) = 0$.

$$d(x + y) = d(x) + d(y) = 0 + 0 = 0,$$

$$\begin{aligned} d(x\alpha y) &= d(x)\alpha f(y) + g(x)\alpha d(y) \\ &= 0\alpha f(y) + g(x)\alpha 0 \\ &= 0 + 0 = 0. \end{aligned}$$

Therefore $x\alpha y$, $x + y \in M$. Hence $\ker d$ is Γ-subsemiring of M.

Suppose $x \leq y$ and $y \in \ker d$. Now, $x + y = y$ implies $d(x + y) = d(y)$, i.e., $d(x) + d(y) = d(y)$, so $d(x) + 0 = 0$. Hence $x \in \ker d$.

Suppose $x + y \in \ker d$, $x \in \ker d$. Then $d(x + y) = 0$ and $d(x) = 0$ then $d(y) = 0$, therefore $y \in \ker d$. Hence $\ker d$ is a $k-$ideal. □

Definition 3.6. An ideal I of an ordered Γ-semiring M is said to be $m-k-$ideal if $x\alpha y \in I$, $x \in I$, $1 \neq y \in M$ and $\alpha \in \Gamma$ then $y \in I$.

Theorem 3.17. *Let d be a $(f, g)-$derivation of an ordered integral Γ-semiring M, $x \leq y$ implies $x + y = y$, for all $x, y \in M$, and g is a non zero endomorphism of M. Then* $\ker d$ *is a $m - k-$ ideal of M.*

Proof. By Theorem 3.16, $\ker d$ is an ideal of M. Suppose $x\alpha y \in \ker d$, $x \in \ker d$, $y \in M$, and $\alpha \in \Gamma$. Then

$$\begin{aligned} d(x\alpha y) &= d(x)\alpha f(y) + g(x)\alpha d(y), \\ 0 &= 0\alpha f(y) + g(x)\alpha d(y), \\ 0 &= g(x)\alpha d(y). \end{aligned}$$

Therefore $d(y) = 0$, since g is a nonzero endomorphism of M, so $y \in \ker d$. Hence $\ker d$ is a $m - k-$ideal of ordered integral Γ-semiring M. □

Theorem 3.18. *Let M be an idempotent cancellative ordered Γ-semiring in which semigroup $(M, +)$ is positively ordered and Γ-semigroup M is negatively ordered. Let d be a (f, g)- derivation of M. Define a set $\{x \in M / f(x) \leq x$ and $g(x) = x)\}$ and let it be denoted by $Fix_d(M)$. Then $Fix_d(M)$ is a $m - k-$ideal of M.*

Proof. Let $x, y \in Fix_d(M)$ and $\alpha \in \Gamma$. Then $f(x) \leq x, g(x) = x, f(y) \leq y$ and $g(y) = y$. Therefore

$$\begin{aligned} f(x\alpha y) &= f(x)\alpha f(y) \leq x\alpha y, \\ g(x\alpha y) &= g(x)\alpha g(y) = x\alpha y. \end{aligned}$$

$$\begin{aligned} f(x + y) &= f(x) + f(y) \leq x + y, \\ g(x + y) &= g(x) + g(y) = x + y. \end{aligned}$$

Therefore $x\alpha y$, $x + y \in Fix_d(M)$.
Suppose $x \leq y$ and $y \in Fix_d(M)$ and $\alpha \in \Gamma$. Then by Corollary 3.1, $d(y) = y$. Now, from $x \leq y$, we have

$$\begin{aligned} x + y \leq y + y = y &\leq x + y, \\ x + y &= y, \\ d(x + y) &= d(y), \\ d(x) + d(y) &= d(y), \\ d(x) + y &= y = x + y, \\ d(x) &= x. \end{aligned}$$

Hence $x \in Fix_d(M)$.

Suppose $x + y \in Fix_d(M)$ and $y \in Fix_d(M)$. Then $d(x + y) = x + y$ and $d(y) = y$. So, $d(x) + d(y) = x + y$, which implies $d(x) + y = x + y$. Therefore $d(x) = x$. Hence $Fix_d(M)$ is a $k-$ideal of M.

Suppose $x\alpha y \in Fix_d(M), \alpha \in \Gamma$ and $x \in Fix_d(M)$. Then

$$\begin{aligned}
&f(x\alpha y) \leq x\alpha y \text{ and } g(x\alpha y) = x\alpha y,\\
&f(x) \leq x \text{ and } g(x) = x,\\
&f(x)\alpha f(y) + x\alpha y = x\alpha y \text{ and } f(x) + x = x,\\
&f(x)\alpha f(y) + (f(x) + x)\alpha y = (f(x) + x)\alpha y,\\
&f(x)\alpha f(y) + f(x)\alpha y + x\alpha y = f(x)\alpha y + x\alpha y,\\
&f(x)\alpha f(y) + f(x)\alpha y = f(x)\alpha y,\\
&f(x)\alpha(f(y) + y) = f(x)\alpha y,\\
&f(x) + y = y.
\end{aligned}$$

Therefore $f(y) \leq y$. Further, we have

$$\begin{aligned}
&g(x\alpha y) = x\alpha y,\\
&g(x)\alpha g(y) = x\alpha y,\\
&x\alpha g(y) = x\alpha y,\\
&g(y) = y.
\end{aligned}$$

Hence $y \in Fix_d(M)$. Thus $Fix_d(M)$ is a $m-k-$ideal of M. □

4. Conclusion

In this paper, we introduced the concept of (f, g)-derivation, which is a generalization of f-derivation and derivation of ordered Γ-semiring and studied some properties of (f, g)-derivation of ordered Γ-semirings. We proved, if d is a (f, g)-derivation of an ordered integra Γ-semiring M then $\ker d$ is a $m-k-$ideal of M and we characterized $m-k-$ideal using (f, g)-derivation of ordered Γ-semiring M.

Acknowledgement

The authors are highly grateful to the referee(s) and editor for their careful reading, valuable suggestions and comments, which helped to improve the presentation of this paper.

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