Strong commutativity preserving derivations on Lie ideals of prime $\Gamma$-rings

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Abstract. Let $M$ be a $\Gamma$-ring and $S \subseteq M$. A mapping $f : M \to M$ is called strong commutativity preserving on $S$ if $[f(x), f(y)]_\alpha = [x, y]_\alpha$, for all $x, y \in S$, $\alpha \in \Gamma$. In the present paper, we investigate the commutativity of the prime $\Gamma$-ring $M$ of characteristic not 2 with center $Z(M) \neq (0)$ admitting a derivation which is strong commutativity preserving on a nonzero square closed Lie ideal $U$ of $M$. Moreover, we also obtain a related result when a mapping $d$ is assumed to be a derivation on $U$ satisfying the condition $d(u) \circ_\alpha d(v) = u \circ_\alpha v$, for all $u, v \in U$, $\alpha \in \Gamma$.

1. Introduction

Nobusawa [13] developed the concept of a gamma ring and then Barnes [1] weakened slightly the defining conditions for a gamma ring. After these definitions a number of mathematicians have studied on gamma rings in the sense of Barnes and Nobusawa and get results parallel to the ring theory (see for example [1], [11], [9]).

Let $R$ be any ring. The symbol $[a, b]$ denotes $ab - ba$ for $a, b \in R$. $R$ is called prime if $aRb = (0)$ implies either $a = 0$ or $b = 0$, and $R$ is called semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $d$ is called a derivation on $R$ if $d(ab) = d(a)b + ad(b)$ holds for all $a, b \in R$.

A mapping $f$ is said to be commutativity preserving on $R$ if $[f(a), f(b)] = 0$ whenever $[a, b] = 0$, for all $a, b \in R$. In 1976, Watkins [14] obtained the first result on commutativity preserving maps for a $n \times n$ matrix algebra when $n \geq 4$ and $f$ is a monomorphism on $R$. Recently, the study of commutativity preserving maps has become an active research area in ring theory (see for example [4], [6], [8], [12] and references therein).

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Let \( S \) be a subset of \( R \). A map \( f \) is called strong commutativity preserving (SCP) on \( S \) if \([f(a), f(b)] = [a, b]\), for all \( a, b \in S \). Clearly, a map that is strong commutativity preserving on a set \( S \) is also commutativity preserving on \( S \), but the inverse is not true in general. The notion of a strong commutativity preserving map was first introduced by H.E. Bell and G. Mason [3]. Later, H.E. Bell and M.N. Daif [2] proved that if a semiprime ring \( R \) admits a nonzero derivation which is strong commutativity preserving on a right ideal \( \rho \) of \( R \), then \( \rho \subseteq Z(R) \) where \( Z(R) \) is the center of \( R \). In particular, \( R \) is commutative if \( \rho = R \). M. Brešar and C.R. Miers [5] characterized SCP additive maps on a semiprime ring. In [10], Brešar and Miers’s result was extended to Lie ideals of prime rings by J.-S. Lin and C.-K. Liu. Later, Q. Deng and M. Ashraf [7] proved that if there exists a derivation \( d \) of a semiprime ring \( R \) and a mapping \( f : I \rightarrow R \) defined on a nonzero ideal \( I \) of \( R \) such that \([f(a), d(b)] = [a, b]\), for all \( a, b \in I \), then \( R \) contains a nonzero central ideal. They also showed that \( R \) is commutative when \( I = R \). There are lots of generalizations similar to these results can be found in the literature.

Recently, X. Xu, J. Ma and Y. Zhou [15] proved that a semiprime \( \Gamma \)-ring with a strong commutativity preserving derivation on itself must be commutative and that a strong commutativity preserving endomorphism \( \sigma \) on a semiprime \( \Gamma \)-ring \( M \) must have the form \( \sigma(a) = a + \xi(a) \) (\( a \in M \)) where \( \xi \) is a map from \( M \) into its center, which extends some results by Bell and Daif to semiprime \( \Gamma \)-rings.

Motivated by all these results, in the present paper, we study strong commutativity preserving derivations on a nonzero square closed Lie ideal of prime \( \Gamma \)-rings and prove that if \( M \) is a prime \( \Gamma \)-ring of characteristic not 2 such that its center \( Z(M) \neq (0) \) and \( d \) is a SCP derivation on a nonzero square closed Lie ideal \( U \) of \( M \), then \( U \subseteq Z(M) \). In particular, \( M \) is commutative if \( U = M \). Moreover, we also obtain the same result when a mapping \( d \) is assumed to be a derivation on \( U \) satisfying the condition \( d(u) \circ_{\alpha} d(v) = u \circ_{\alpha} v \), for all \( u, v \in U, \alpha \in \Gamma \).  

2. Preliminaries

Before giving our results, we first present some preliminary definitions. In this paper, \( M \) will represent a \( \Gamma \)-ring in the sense of Barnes [1] unless otherwise stated.

An additive subgroup \( K \) of a \( \Gamma \)-ring \( M \) is called a left (resp. right) ideal of \( M \) if \( \Gamma K \subseteq K \) (resp. \( K \Gamma \subseteq K \)). A left ideal \( K \) of a \( \Gamma \)-ring \( M \) is called an ideal of \( M \) if it is also a right ideal of \( M \). The set of all elements \( a \) satisfying \( aab = boa \) for all \( b \in M \) and \( \alpha \in \Gamma \) is called the center of \( M \).

A \( \Gamma \)-ring \( M \) is said to be prime if \( a \Gamma M \Gamma b = (0) \) for \( a, b \in M \) implies that \( a = 0 \) or \( b = 0 \). An additive mapping \( d \) is called a derivation on \( M \) if \( d(aab) = d(a)ab + aad(b) \), for all \( a, b \in M \) and \( \alpha \in \Gamma \).
Let $M$ be a $\Gamma$-ring and $a, b \in M$, $\alpha \in \Gamma$. The commutator of $a$ and $b$ with respect to $\alpha$ is defined as the element $a\alpha b - b\alpha a$ and denoted by $[a, b]_\alpha$. According to this definition we have the following equations,

\begin{equation}
[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha [b, c]_\beta + a\alpha c\beta b - a\beta c\alpha b,
\end{equation}

\begin{equation}
[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha [a, c]_\beta + b\beta a\alpha c - b\alpha a\beta c,
\end{equation}

where $a, b, c \in M$, $\alpha, \beta \in \Gamma$. Similarly, the anti-commutator of $a$ and $b$ with respect to $\alpha$ is defined as the element $a\alpha b + b\alpha a$ and denoted by $a \circ_\alpha b$. According to this definition we have the following equations,

\begin{equation}
(a\alpha b) \circ_\beta c = a\alpha (b \circ_\beta c) - [a, c]_\beta \alpha b + a\alpha c\beta b - a\beta c\alpha b
\end{equation}

\begin{equation}
= (a \circ_\beta c)\alpha b + a\alpha [b, c]_\beta + a\beta c\alpha b - a\alpha c\beta b,
\end{equation}

\begin{equation}
a \circ_\beta (b\alpha c) = (a \circ_\beta b)\alpha c - b\alpha [a, c]_\beta + b\beta a\alpha c - b\alpha a\beta c
\end{equation}

\begin{equation}
= b\alpha (a \circ_\beta c) + [a, b]_\beta \alpha c + b\alpha a\beta c - b\beta a\alpha c,
\end{equation}

where $a, b, c \in M$, $\alpha, \beta \in \Gamma$.

An additive subgroup $U$ of a $\Gamma$-ring $M$ is called a Lie ideal if $[u, m]_\alpha \in U$, for all $u \in U$, $m \in M$ and $\alpha \in \Gamma$. A Lie ideal $U$ of $M$ is said to be a square closed Lie ideal of $M$, if $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. Clearly, $u\alpha v + v\alpha u \in U$, for all $u, v \in U$, $\alpha \in \Gamma$. Similarly, we have $u\alpha v - v\alpha u \in U$.

Moreover, by using these relations, we get $2u\alpha v \in U$ which will be used in the whole paper frequently.

A map $f$ from a $\Gamma$-ring $M$ into itself is called strong commutativity preserving (SCP) on a subset $S$ of $M$ if $[f(a), f(b)]_\alpha = [a, b]_\alpha$ holds for all $a, b \in S$ and $\alpha \in \Gamma$.

3. The Results

First, we work on SCP derivations on Lie ideals of prime $\Gamma$-rings. The following lemma will play an crucial role in the proofs of our main theorems.

**Lemma 3.1.** Let $M$ be a prime $\Gamma$-ring and $Z(M) \neq (0)$. Then the equations

\begin{equation}
[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha [b, c]_\beta,
\end{equation}

\begin{equation}
[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha [a, c]_\beta
\end{equation}

hold for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$.

**Proof.** For any $c \in M$, $\alpha, \beta \in \Gamma$, the symbol $[\alpha, \beta]_c$ denotes $\alpha c\beta - \beta c\alpha$. Then, the commutator formulas in (1) and (2) become

\begin{equation}
[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha [b, c]_\beta + a[\alpha, \beta]_c b
\end{equation}

and

\begin{equation}
[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha [a, c]_\beta + b[\beta, \alpha]_c a c
\end{equation}

for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$. 
Since \( Z(M) \neq (0) \), there exists a nonzero element \( x \) in \( Z(M) \). Thus,
\[
\begin{align*}
  x \gamma y \delta a c \beta b & = y \gamma x \delta a c \beta b = y \gamma a \delta x a c \beta b \\
  & = y \gamma a \delta c a x \beta b = y \gamma a \delta c a b \beta x \\
  & = y \gamma a \delta x \beta c a b = y \gamma x \delta a \beta c a b \\
  & = x \gamma y \delta a \beta c a b,
\end{align*}
\]
for all \( a, b, c, y \in M, \alpha, \beta, \gamma, \delta \in \Gamma \). Then we have that
\[
(4) \quad x \gamma y \delta a [\alpha, \beta] c b = 0,
\]
for all \( a, b, c, y \in M, \alpha, \beta, \gamma, \delta \in \Gamma \). Multiplying the two sides of (3) by \( x \gamma y \delta \) from the left hand side, and then comparing with (4) we get for all \( a, b, c, y \in M \), \( \alpha, \beta, \gamma, \delta \in \Gamma \)
\[
x \gamma y \delta [a c b, \beta] = x \gamma y \delta [a, c] \beta a b + x \gamma y \delta a a [b, c] \beta.
\]
That is \( x \Gamma M \Gamma ([a c b, \beta] - [a, c] \beta a b - a a [b, c] \beta) = 0 \), for all \( a, b, c \in M \), \( \alpha, \beta \in \Gamma \). Since \( M \) is prime and \( x \) is nonzero, we have
\[
[a c b, \beta] - [a, c] \beta a b - a a [b, c] \beta = 0,
\]
for all \( a, b, c \in M \), \( \alpha, \beta \in \Gamma \). For the second equation, one can use the same method above, and this completes the proof. \( \square \)

Now, we can give a similar result for the anti-commutator formulas of \( \Gamma \)-rings.

**Lemma 3.2.** Let \( M \) be a prime \( \Gamma \)-ring in the sense of Barnes and \( Z(M) \neq (0) \). Then the equations
\[
(aa b) \circ_\beta c = a a (b \circ_\beta c) - [a, c] \beta a b
\]
hold for all \( a, b, c \in M, \alpha, \beta \in \Gamma \).

**Proof.** It can be proved by using the techniques of Lemma 3.1. \( \square \)

We need the following results to prove our main theorems.

**Lemma 3.3.** Let \( M \) be a prime \( \Gamma \)-ring of characteristic not 2 with the center \( Z(M) \neq (0) \) and \( U \) be a Lie ideal of \( M \). If \( U \not\subseteq Z(M) \), then there exists an ideal \( K \) of \( M \) such that \( [K, M] \Gamma \subseteq U \) but \( [K, M] \Gamma \not\subseteq Z(M) \).

**Proof.** First, we show that the Lie product of \( U \) by itself is different from zero. Suppose that \( [U, U] \Gamma = (0) \). Then we have \( [a, [a, m] \alpha] \beta = 0 \), for all \( a \in U, m \in M \) and \( \alpha, \beta \in \Gamma \). Replacing \( m \) by \( m \gamma x \) for \( \gamma \in \Gamma \) and \( x \in M \), we get
\[
(5) \quad [a, m] \beta \gamma [a, x] \alpha + [a, m] \alpha \gamma [a, x] \beta = 0.
\]
Now, replacing $\beta$ by $\alpha$ in (5) we have $[a, m]_\alpha \gamma [a, x]_\alpha = 0$, for all $a \in U$, $m, x \in M$ and $\alpha, \gamma \in \Gamma$. Replacing $x$ by $y \delta x$ for $y \in M$ and $\delta \in \Gamma$ in the last equation, we get $[a, m]_\alpha \Gamma M T [a, x]_\alpha = (0)$, for all $a \in U$, $m, x \in M$ and $\alpha \in \Gamma$. Therefore, we have $U \subseteq Z(M)$ since $M$ is prime. But this contradicts with the hypothesis of the theorem. Hence, there exist $u, v \in U$ and $\beta \in \Gamma$ such that $[u, v]_\beta \neq 0$.

Let $K := M \Gamma [u, v]_\beta \Gamma M$ and $T(U) := \{x \in M \mid [x, M]_\Gamma \subseteq U\}$. Then, it is clear that $K \neq (0)$ is an ideal of $M; T(U)$ is a Lie ideal and a subring of $M$. Moreover, $U \subseteq T(U)$. Since $[u, v]_{\gamma m} = [u, v]_\beta \gamma m + v \gamma [u, m]_\beta$ for all $m \in M$ and $\gamma \in \Gamma$, we get $[u, v]_\beta \Gamma M \subseteq T(U)$. Hence,

$$\left[[u, v]_\beta \alpha m, n\right]_\gamma \in T(U),$$

for all $n, m \in M$ and $\alpha, \gamma \in \Gamma$. Expanding this we get $n \gamma [u, v]_\beta \alpha m \in T(U)$ for all $n, m \in M$ and $\alpha, \gamma \in \Gamma$. Then, we have $M \Gamma [u, v]_\beta \Gamma M = K \subseteq T(U)$ which yields to $[K, M]_\Gamma \subseteq U$.

Now, suppose $[K, M]_\Gamma \subseteq Z(M)$. Therefore, we have $[K, [K, M]_\Gamma]_\Gamma = (0)$ and using the same argument above we get $K \subseteq Z(M)$. Let $x \in M$. Then $n \alpha k \gamma m \in K$ for all $n, m \in M$, $k \in K$ and $\alpha, \gamma \in \Gamma$. Since $K \subseteq Z(M)$ we have $[x, n \alpha k \gamma m]_\beta = 0$. Expanding this we get $K \Gamma M \Gamma [x, M]_\Gamma = (0)$. Therefore, $x \in Z(M)$ since $M$ is prime and $K \neq (0)$. But this contradicts with $U \not\subseteq Z(M)$. This completes the proof. \hfill \Box

Lemma 3.4. Let $M$ be a prime $\Gamma$-ring of characteristic not 2 with the center $Z(M) \neq (0)$ and $U$ be a Lie ideal of $M$. If $U \not\subseteq Z(M)$ and $a, b \in M$ such that $a \Gamma U \Gamma b = (0)$, then either $a = 0$ or $b = 0$.

Proof. By Lemma 3.3, there exists an ideal $K$ of $M$ such that $[K, M]_\Gamma \subseteq U$ but $[K, M]_\Gamma \not\subseteq Z(M)$ . Let $u \in U$, $k \in K$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Then, we have

$$[k \alpha \alpha \beta u, m]_\gamma \in [K, M]_\Gamma \subseteq U.$$ 

It follows from that

$$0 = a \lambda [k \alpha \alpha \beta u, m]_\gamma e b = a \lambda k \alpha \alpha \beta [u, m]_\gamma e b + a \lambda [k \alpha \alpha a, m]_\gamma \beta u e b$$

$$= a \lambda k \alpha \alpha a \gamma m \beta u e b - a \lambda m \gamma k \alpha \alpha \beta u e b$$

$$= a \lambda k \alpha \alpha a \gamma m \beta u e b,$$

for all $u \in U$, $k \in K$, $m \in M$ and $\alpha, \beta, \gamma, \lambda, \epsilon \in \Gamma$. Therefore, we get $a \Gamma K \Gamma a = (0)$ or $U \Gamma b = (0)$ since $M$ is prime. In the first case, we see that $a$ must be zero by using the primeness of $M$. In the second case, we get

$$[u, m]_\alpha \gamma b = 0,$$

for all $u \in U$, $m \in M$ and $\alpha, \gamma \in \Gamma$. Expanding this we have

$$[u \gamma b, m]_\alpha - u \gamma [b, m]_\alpha = 0,$$
that is \( u\gamma mab = 0 \), for all \( u \in U, m \in M \) and \( \alpha, \gamma \in \Gamma \). Therefore, \( b = 0 \) since \( M \) is prime and \( U \neq (0) \).

**Lemma 3.5.** Let \( M \) be a prime \( \Gamma \)-ring with the center \( Z(M) \neq (0) \) and \( x \in M \). If \( a \in Z(M) \) and \( a\gamma x \in Z(M) \) for all \( \gamma \in \Gamma \), then \( a = 0 \) or \( x \in Z(M) \).

**Proof.** Suppose that \( a \neq 0 \). Since \( a\gamma x \in Z(M) \), we have \([a\gamma x, m]_{\delta} = 0\) for all \( m \in M \) and \( \delta, \gamma \in \Gamma \). Expanding this we get \( a\gamma [x, m]_{\delta} = 0 \). Replacing \( m \) by \( m\beta n \) for \( n \in M \) and \( \beta \in \Gamma \) we conclude that \( x \in Z(M) \) since \( M \) is prime. This completes the proof. \( \square \)

**Lemma 3.6.** Let \( M \) be a prime \( \Gamma \)-ring of characteristic not 2 with the center \( Z(M) \neq (0) \) and \( U \) be a Lie ideal of \( M \). If \([U, U]_{\Gamma} \subseteq Z(M)\), then \( U \subseteq Z(M) \).

**Proof.** By hypothesis we have \([u, [u, x]_{\alpha}]_{\beta} \in Z(M)\) for all \( u \in U, x \in M \) and \( \alpha, \beta \in \Gamma \). Since

\[
[u, [u, x]_{\alpha}]_{\beta} \gamma u = [u, [u, x]_{\alpha} \gamma u]_{\beta} = [u, [u, x\gamma u]_{\alpha}]_{\beta}
\]

and \([u, [u, x\gamma u]_{\alpha}]_{\beta} \in [U, U]_{\Gamma}\), we have \([u, [u, x]_{\alpha}]_{\beta} \gamma u \in Z(M)\). Therefore, we get \([u, [u, x]_{\alpha}]_{\beta} = 0\) or \( u \in Z(M) \) by Lemma 3.5. Now, let \([u, [u, x]_{\alpha}]_{\beta} = 0\) for all \( x \in M, \alpha, \beta \in \Gamma \) and for some \( u \in U \). Replacing \( x \) by \( x\gamma m \) we get

\[
[u, x]_{\beta} \gamma [u, m]_{\alpha} + [u, x]_{\alpha} \gamma [u, m]_{\beta} = 0,
\]

for all \( x, m \in M \) and \( \alpha, \beta, \gamma \in \Gamma \). Replacing \( \beta \) by \( \alpha \) in the equation (6) we get \([u, x]_{\alpha} \gamma [u, m]_{\alpha} = 0\) since \( M \) is a \( \Gamma \)-ring of characteristic not 2. Replacing \( m \) by \( m\delta n \) for \( n \in M, \delta \in \Gamma \) in the last equation, we conclude that \( u \in Z(M) \) since \( M \) is prime. Consequently, we see that \( U \) must be a subset of \( Z(M) \). \( \square \)

**Theorem 3.1.** Let \( M \) be a prime \( \Gamma \)-ring of characteristic not 2 and \( U \) be a nonzero square closed Lie ideal of \( M \). If \( d \) is a SCP derivation on \( U \), then \( U \subseteq Z(M) \) or \( Z(M) = (0) \).

**Proof.** Suppose that \( Z(M) \neq (0) \). We have \([d(x), d(y)]_{\alpha} = [x, y]_{\alpha}\) for all \( x, y \in U \) and \( \alpha \in \Gamma \) by hypothesis. Replacing \( y \) by \( 2y\beta z \) for \( z \in U \) and \( \beta \in \Gamma \), we get

\[
[d(x), d(2y\beta z)]_{\alpha} = [x, 2y\beta z]_{\alpha},
\]

for all \( x, y, z \in U \) and \( \alpha, \beta \in \Gamma \). By applying Lemma 3.1, we expand the last equation and we get

\[
d(y)\beta [d(x), z]_{\alpha} + [d(x), y]_{\alpha} \beta d(z) = 0,
\]

since \( M \) is a \( \Gamma \)-ring of characteristic not 2. Replacing \( z \) by \( 2z\gamma t \) for \( z, t \in U \) and \( \gamma \in \Gamma \) in the equation (7) we obtain that

\[
d(y)\beta [d(x), z]_{\alpha} \gamma t + d(y)\beta z\gamma [d(x), t]_{\alpha} + [d(x), y]_{\alpha} \beta d(z) \gamma t + [d(x), y]_{\alpha} \beta z \gamma d(t) = 0,
\]

(8)
since $M$ is a $\Gamma$-ring of characteristic not 2. Multiplying the two sides of (7) by $\gamma t$ from the right hand side, we have

\[(9) \quad d(y)\beta[d(x),z]_\alpha \gamma t + [d(x),y]_\alpha \beta d(z)\gamma t = 0,
\]
for all $x, y, z, t \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Comparing (9) with (8), we have that

\[d(y)\beta z\gamma[d(x),t]_\alpha + [d(x),y]_\alpha \beta z\gamma d(t) = 0,
\]
for all $x, y, z, t \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Since $U$ is a nonzero square closed Lie ideal of $M$, we have $[U,U]_\Gamma$ is a nonzero square closed Lie ideal of $M$, too. Writing $t = d(x)$ for $x \in [U,U]_\Gamma$, we obtain that

\[(10) \quad [d(x),y]_\alpha \beta z\gamma d^2(x) = 0,
\]
for all $y, z \in U$, $x \in [U,U]_\Gamma$ and $\alpha, \beta, \gamma \in \Gamma$. If we replace $y$ by $d(y)$ for $y \in [U,U]_\Gamma$ in the equation (10), we obtain $[x,y]_\alpha \Gamma U \Gamma d^2(x) = (0)$ for all $x, y \in [U,U]_\Gamma$, and $\alpha \in \Gamma$ since $d$ is SCP on $U$. Therefore,

\[ [x,y]_\alpha \beta 2[m,z]_\alpha \Gamma U \Gamma [x,y]_\alpha \beta 2[m,z]_\alpha = (0),
\]
since

\[ [x,y]_\alpha \Gamma U \Gamma [d^2(x),d^2(y)]_\alpha \beta 2[m,z]_\alpha = (0),
\]
for all $x, y \in [U,U]_\Gamma$, $m \in M$, $z \in U$ and $\alpha, \beta \in \Gamma$. Since $M$ is a $\Gamma$-ring of characteristic not 2, we have $[x,y]_\alpha \beta [m,z]_\alpha = 0$ by Lemma 3.4. Replacing $m$ by $m\gamma t$ for $t \in M$ and $\gamma \in \Gamma$ we get

\[ [x,y]_\alpha \beta m\gamma [t,z]_\alpha = 0,
\]
for all $x, y \in [U,U]_\Gamma$, $m, t \in M$, $z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. By the primeness of the $\Gamma$-ring $M$, we get either $[x,y]_\alpha = 0$ or $[t,z]_\alpha = 0$, for all $x, y \in [U,U]_\Gamma$, $z \in U$, $t \in M$ and $\alpha \in \Gamma$. In the second case, we see that $z \in Z(M)$ that is $U \subseteq Z(M)$. In the first case, using Lemma 3.6, we have $[U,U]_\Gamma \subseteq Z(M)$. Consequently, applying Lemma 3.6 again, we get that $U \subseteq Z(M)$ which completes the proof. \[\square\]

In particular, if we take $U = M$, then Theorem 3.1 gives a commutativity criterion as follows.

**Corollary 3.1.** Let $M$ be a prime $\Gamma$-ring of characteristic not 2 and $d$ be a derivation of $M$. If $Z(M) \neq (0)$ and $d$ is SCP on $M$, then $M$ is commutative.

Since we can use the similar techniques of Theorem 3.1, we can obtain the following theorems which partially generalize the result of Bell and Daif to prime $\Gamma$-rings.

**Theorem 3.2.** Let $M$ be a prime $\Gamma$-ring of characteristic not 2 and $U$ be a nonzero square closed Lie ideal of $M$. If $[d(x),d(y)]_\alpha = -[x,y]_\alpha$ for all $x,y \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z(M)$ or $Z(M) = (0)$.

**Proof.** It can be proved easily by using the same method in Theorem 3.1. \[\square\]
Corollary 3.2. Let $M$ be a prime $\Gamma$-ring of characteristic not 2 and $d$ be a derivation of $M$. If $Z(M) \neq (0)$ and $[d(x), d(y)]_\alpha = -[x, y]_\alpha$ for all $x, y \in M$, $\alpha \in \Gamma$, then $M$ is commutative.

Theorem 3.3. Let $M$ be a prime $\Gamma$-ring of characteristic not 2 and $U$ be a nonzero square closed Lie ideal of $M$. If $d$ is a derivation of $M$ such that $d(x) \circ_\alpha d(y) = x \circ_\alpha y$ for all $x, y \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z(M)$ or $Z(M) = (0)$.

Proof. Suppose that $Z(M) \neq (0)$. By the hypothesis we obtain that

$$d(x) \circ_\alpha d(y) - x \circ_\alpha y = 0,$$

for all $x, y \in U$ and $\alpha \in \Gamma$. Replacing $x$ by $2x\beta z$ for $z \in U$, $\beta \in \Gamma$ in the equation (11) we get

$$d(x)\beta[z, d(y)]_\alpha - [x, d(y)]_\alpha \beta d(z) + 2x\beta y\alpha z = 0,$$

since $M$ is a $\Gamma$-ring of characteristic not 2. Taking $2z\gamma x$ for $z$ in the equation (12) we have

$$d(x)\beta[z, d(y)]_\alpha \gamma x + d(x)\beta z\gamma[x, d(y)]_\alpha - [x, d(y)]_\alpha \beta d(z)\gamma x
- [x, d(y)]_\alpha \beta z\gamma d(x) + 2x\beta y\alpha z\gamma x = 0,$$

for all $x, y, z \in U$, $\alpha, \beta, \gamma \in \Gamma$. Multiplying the two sides of (12) by $\gamma x$ from the right hand side, we get

$$d(x)\beta[z, d(y)]_\alpha \gamma x - [x, d(y)]_\alpha \beta d(z)\gamma x + 2x\beta y\alpha z\gamma x = 0,$$

for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. If we compare (13) and (14), we have that

$$d(x)\beta z\gamma[x, d(y)]_\alpha - [x, d(y)]_\alpha \beta z\gamma d(x) = 0,$$

for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Replacing $z$ by $2z\sigma[x, d(y)]_\alpha$ for $y \in [U, U]_\Gamma$ and $\sigma \in \Gamma$ in the equation (15) we get

$$d(x)\beta z\sigma[x, d(y)]_\alpha \gamma[x, d(y)]_\alpha - [x, d(y)]_\alpha \beta z\sigma[x, d(y)]_\alpha \gamma d(x) = 0,$$

since $M$ is a $\Gamma$-ring of characteristic not 2. Taking $\sigma$ for $\gamma$ in (15) we have

$$d(x)\beta z\sigma[x, d(y)]_\alpha = [x, d(y)]_\alpha \beta z\sigma d(x),$$

If we use the equation (17) in the equation (16) we get

$$[x, d(y)]_\alpha \beta z\sigma d(x)\gamma[x, d(y)]_\alpha = [x, d(y)]_\alpha \beta z\sigma[x, d(y)]_\alpha \gamma d(x)$$
and so

$$[x, d(y)]_\alpha \beta z\sigma[d(x), [x, d(y)]_\alpha]_\gamma = 0,$$
for all $x, z \in U$, $y \in [U, U]_\Gamma$ and $\alpha, \beta, \gamma, \sigma \in \Gamma$. Taking $\beta = \gamma$ in (18), we get

$$[x, d(y)]_\alpha \gamma z\sigma[d(x), [x, d(y)]_\alpha]_\gamma = 0,$$
for all $x, z \in U$, $y \in [U, U]_{\Gamma}$ and $\alpha, \gamma, \sigma \in \Gamma$. Multiplying the equation (19) on the left by $d(x)\gamma$ for $x \in [U, U]_{\Gamma}$, we have

\begin{equation}
(20) \quad d(x)\gamma[x, d(y)]_{\alpha}\gamma z \sigma d(x), [x, d(y)]_{\alpha} \gamma = 0.
\end{equation}

Taking $2d(x)\gamma z$ for $z$ in (19) we obtain that

\begin{equation}
(21) \quad [x, d(y)]_{\alpha}\gamma d(x)\gamma z \sigma d(x), [x, d(y)]_{\alpha} \gamma = 0,
\end{equation}

for all $z \in U$, $x, y \in [U, U]_{\Gamma}$ and $\alpha, \gamma, \sigma \in \Gamma$ since $M$ is a $\Gamma$-ring of characteristic not 2. Subtracting (21) from (20) we see that

\begin{equation}
[d(x), [x, d(y)]_{\alpha}]\gamma z \sigma d(x), [x, d(y)]_{\alpha} \gamma = 0,
\end{equation}

for all $z \in U$, $x, y \in [U, U]_{\Gamma}$ and $\alpha, \gamma, \sigma \in \Gamma$. Therefore, by Lemma 3.4 we have that

\begin{equation}
(22) \quad [d(x), [x, d(y)]_{\alpha}] \gamma = 0,
\end{equation}

for all $x, y \in [U, U]_{\Gamma}$ and $\alpha, \gamma \in \Gamma$. Replacing $z$ by $x$ for $x \in [U, U]_{\Gamma}$ and $\beta = \gamma$ in (12) and using the equation (22) we conclude that $x\Gamma[U, U]_{\Gamma} \Gamma x = (0)$ for all $x \in [U, U]_{\Gamma}$ since $M$ is a $\Gamma$-ring of characteristic not 2. We know that $[U, U]_{\Gamma}$ is a nonzero square closed Lie ideal of $M$. So by using Lemma 3.4 we get either $x = 0$ for all $x \in [U, U]_{\Gamma}$ or $[U, U]_{\Gamma} \subseteq Z(M)$. The first case contradicts with the hypothesis $[U, U]_{\Gamma} \neq (0)$. Then we have that $[U, U]_{\Gamma} \subseteq Z(M)$. Hence, applying Lemma 3.6 we obtain that $U \subseteq Z(M)$. This completes the proof. \qed

**Corollary 3.3.** Let $d$ be a derivation of a prime $\Gamma$-ring $M$ of characteristic not 2. If $d(x) \circ_{\alpha} d(y) = x \circ_{\alpha} y$ for all $x, y \in M$, $\alpha \in \Gamma$ and $Z(M) \neq (0)$, then $M$ is commutative.

**Theorem 3.4.** Let $M$ be a prime $\Gamma$-ring of characteristic not 2 and $U$ be a nonzero square closed Lie ideal of $M$. If $d$ is a derivation of $M$ such that $d(x) \circ_{\alpha} d(y) = -(x \circ_{\alpha} y)$ for all $x, y \in U$, $\alpha \in \Gamma$, then $U \subseteq Z(M)$ or $Z(M) = (0)$.

**Proof.** Suppose that $Z(M) \neq (0)$. By the hypothesis we have that

\begin{equation}
(23) \quad d(x) \circ_{\alpha} d(y) + x \circ_{\alpha} y = 0,
\end{equation}

for all $x, y \in U$ and $\alpha \in \Gamma$. Replacing $x$ by $2x\beta z$ for $z \in U$, $\beta \in \Gamma$ in the equation (23) we get

\begin{equation}
(24) \quad d(x)\beta [z, d(y)]_{\alpha} - [x, d(y)]_{\alpha} \beta d(z) + 2x\beta z\alpha y = 0,
\end{equation}

since $M$ is a $\Gamma$-ring of characteristic not 2. Taking $2z\gamma x$ for $x$ in the equation (24) we have

\begin{equation}
(25) \quad d(z)\gamma x\beta [z, d(y)]_{\alpha} + z\gamma d(x)\beta [z, d(y)]_{\alpha} - [z, d(y)]_{\alpha} \gamma x\beta d(z) - z\gamma [x, d(y)]_{\alpha} \beta d(z) + 2z\gamma x\beta z\alpha y = 0.
\end{equation}

Multiplying the two sides of (24) by $z\gamma$ from the left hand side, we get

\begin{equation}
(26) \quad z\gamma d(x)\beta [z, d(y)]_{\alpha} - z\gamma [x, d(y)]_{\alpha} \beta d(z) + 2z\gamma x\beta z\alpha y = 0,
\end{equation}

This completes the proof.
for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. If we compare (25) and (26), we have that

$$d(z)\gamma x \beta[z, d(y)]_\alpha - [z, d(y)]_\alpha \gamma x \beta d(z) = 0,$$

for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Replacing $x$ by $2x\sigma[z, d(y)]_\alpha$ for $y \in [U, U]_\Gamma$ and $\sigma \in \Gamma$ in the equation (27) we get

$$d(z)\gamma x \sigma[z, d(y)]_\alpha \beta[z, d(y)]_\alpha - [z, d(y)]_\alpha \gamma x \sigma[z, d(y)]_\alpha \beta d(z) = 0,$$

since $M$ is a $\Gamma$-ring of characteristic not 2. Taking $\sigma$ for $\beta$ in (27) we have

$$d(z)\gamma x \sigma[z, d(y)]_\alpha = [z, d(y)]_\alpha \gamma x \sigma d(z).$$

If we use the equation (29) in the equation (28) we get

$$[z, d(y)]_\alpha \gamma x \sigma d(z)\beta[z, d(y)]_\alpha = [z, d(y)]_\alpha \gamma x \sigma[z, d(y)]_\alpha \beta d(z)$$

and so

$$[z, d(y)]_\alpha \gamma x \sigma[d(z)], [z, d(y)]_\alpha \beta = 0,$$

for all $x, z \in U$, $y \in [U, U]_\Gamma$ and $\alpha, \beta, \gamma, \sigma \in \Gamma$. Taking $\beta = \gamma$ in (30), we get

$$[z, d(y)]_\alpha \gamma x \sigma[d(z)], [z, d(y)]_\alpha \gamma = 0,$$

for all $x, z \in U$, $y \in [U, U]_\Gamma$ and $\alpha, \gamma, \sigma \in \Gamma$. Multiplying the equation (31) on the left by $d(z)\gamma$ for $z \in [U, U]_\Gamma$, we have

$$d(z)\gamma[z, d(y)]_\alpha \gamma x \sigma[d(z)], [z, d(y)]_\alpha \gamma = 0.$$

Taking $2d(z)\gamma x$ for $x$ in (31) we obtain that

$$[z, d(y)]_\alpha \gamma d(z)\gamma x \sigma[d(z)], [z, d(y)]_\alpha \gamma = 0,$$

for all $x \in U$, $y, z \in [U, U]_\Gamma$ and $\alpha, \gamma, \sigma \in \Gamma$ since $M$ is a $\Gamma$-ring of characteristic not 2. Subtracting (33) from (32) we see that

$$[d(z), [z, d(y)]_\alpha \gamma x \sigma[d(z)], [z, d(y)]_\alpha \gamma = 0,$$

for all $x \in U$, $y, z \in [U, U]_\Gamma$ and $\alpha, \gamma, \sigma \in \Gamma$. Therefore, by Lemma 3.4 we have that

$$[d(z), [z, d(y)]_\alpha \gamma = 0,$$

for all $y, z \in [U, U]_\Gamma$ and $\alpha, \gamma \in \Gamma$. Then, the proof is completed by using the similar steps in the equation (22) in Theorem 3.3.

**Corollary 3.4.** Let $d$ be a derivation of a prime $\Gamma$-ring $M$ of characteristic not 2. If $d(x)\circ_\alpha d(y) = -(x \circ_\alpha y)$ for all $x, y \in M$, $\alpha \in \Gamma$ and $Z(M) \neq (0)$, then $M$ is commutative.

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