New inequalities for \( F \)-convex functions pertaining generalized fractional integrals

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Abstract. In this paper, the authors, utilizing \( F \)-convex functions which are defined by B. Samet, establish some new Hermite-Hadamard type inequalities via generalized fractional integrals. Some special cases of our main results recaptured the well-known earlier works.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). If \( f \) is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [17]:

\[
(1) \quad f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Both inequalities in (1) hold in the reversed direction if \( f \) is concave.

Over the last decade, this classical double inequality has been improved and generalized in a number of ways, see [5, 7, 8, 13, 18], [23]–[25] and the references therein. Also, many types of convexities have been defined, such as quasi–convex in [6], pseudo–convex in [14], strongly convex in [20], \( \varepsilon \)-convex in [11], \( s \)-convex in [10], \( h \)-convex in [28], etc. Recently, Samet in [21], has defined a new concept of convexity that depends on a certain function satisfying some axioms, that generalizes different types of convexity.

Recall the family \( \mathcal{F} \) of mappings \( F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \) satisfying the following axioms:

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(A1) If \( e_i \in L^1(0,1), \ i = 1, 2, 3, \) then for every \( \lambda \in [0,1], \) we have

\[
\int_0^1 F(e_1(t), e_2(t), e_3(t), \lambda)dt = F\left( \int_0^1 e_1(t)dt, \int_0^1 e_2(t)dt, \int_0^1 e_3(t)dt, \lambda \right);
\]

(A2) For every \( u \in L^1(0,1), \ w \in L^\infty(0,1) \) and \((z_1, z_2) \in \mathbb{R}^2,\) we have

\[
\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, 2)dt = T_{F,w}\left( \int_0^1 w(t)u(t)dt, z_1, z_2 \right),
\]

where \( T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a function that depends on \((F, w),\)

and it is nondecreasing with respect to the first variable;

(A3) For any \((w, e_1, e_2, e_3) \in \mathbb{R}^4, \ e_4 \in [0,1],\) we have

\[
wF(e_1, e_2, e_3, e_4) = w(e_1, we_2, we_3, e_4) + L_w,
\]

where \( L_w \in \mathbb{R} \) is a constant that depends only on \( w.\)

**Definition 1.** Let \( f : [a, b] \to \mathbb{R}, \ (a, b) \in \mathbb{R}^2, \ a < b, \) be a given function. We say that \( f \) is a convex function with respect to some \( F \in \mathcal{F} \) (or \( F \)-convex function), if and only if:

\[
F(f(tx + (1-t)y), f(x), f(y), t) \leq 0 , \ (x, y, t) \in [a, b] \times [a, b] \times [0,1].
\]

**Remark 1.** 1) Let \( \varepsilon \geq 0, \) and let \( f : [a, b] \to \mathbb{R}, \ (a, b) \in \mathbb{R}^2, \ a < b, \) be an \( \varepsilon \)-convex function, see [11], that is

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon , \ (x, y, t) \in [a, b] \times [a, b] \times [0,1].
\]

Define the functions

\[
F(e_1, e_2, e_3, e_4) = e_1 - e_4 e_2 - (1-e_4)e_3 - \varepsilon,
\]

and \( T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by

\[
T_{F,w}(e_1, e_2, e_3) = e_1 - \left( \int_0^1 t w(t)dt \right) e_2 - \left( \int_0^1 (1-t) w(t)dt \right) e_3 - \varepsilon.
\]

For

\[
L_w = (1-w)\varepsilon,
\]

it is clear that \( F \in \mathcal{F} \) and

\[
F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \leq 0,
\]

that is \( f \) is an \( F \)-convex function. Particularly, taking \( \varepsilon = 0, \) we show that if \( f \) is a convex function then \( f \) is an \( F \)-convex function with respect to \( F \) defined above.
2) Let \( h : J \rightarrow [0, +\infty) \) be a given function which is not identical to 0, where \( J \) is an interval in \( \mathbb{R} \) such that \((0, 1) \subseteq J\). Let \( f : [a, b] \rightarrow [0, +\infty) \), \((a, b) \in \mathbb{R}^2\), \( a < b \), be an \( h \)-convex function, see [28], that is

\[
 f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y), \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].
\]

Define the functions \( F : \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \) by

\[
 F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1 - e_4)e_3
\]

and \( T_{F, w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) by

\[
 T_{F, w}(e_1, e_2, e_3) = e_1 - \left( \int_0^1 h(t)w(t)dt \right) e_2 - \left( \int_0^1 h(1-t)w(t)dt \right) e_3.
\]

For \( L_w = 0 \), it is clear that \( F \in \mathcal{F} \) and

\[
 F(f(tx+(1-t)y), f(x), f(y), t) = f(tx+(1-t)y) - h(t)f(x) - h(1-t)f(y) \leq 0,
\]

that is, \( f \) is an \( F \)-convex function.

Samet in [21], established the following Hermite–Hadamard type inequalities using the new convexity concept:

**Theorem 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \), \((a, b) \in \mathbb{R}^2\), \( a < b \), be an \( F \)-convex function, for some \( F \in \mathcal{F} \). Suppose that \( f \in L^1[a, b] \). Then

\[
 F \left( f \left( \frac{a+b}{2} \right), \frac{1}{b-a} \int_a^b f(x)dx, \frac{1}{b-a} \int_a^b f(x)dx, \frac{1}{2} \right) \leq 0,
\]

\[
 T_{F, 1} \left( \frac{1}{b-a} \int_a^b f(x)dx, f(a), f(b) \right) \leq 0.
\]

**Definition 2.** Let \( f \in L^1[a, b] \). The Riemann–Liouville integrals \( J_{a+}^\alpha f \) and \( J_{b-}^\alpha f \) of order \( \alpha > 0 \) are defined by

\[
 J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
\]

and

\[
 J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,
\]

respectively. Here, \( \Gamma(\alpha) \) is the Gamma function and

\[
 J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x).
\]

**Definition 3.** Let \( f \in L^1[a, b] \). Then \( k \)-fractional integrals of order \( \alpha, k > 0 \) are defined by

\[
 J_{a+}^{\alpha, k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a,
\]
New inequalities for $F$-convex functions

and

\begin{equation}
I_{b^+,k}^\alpha f(x) = \frac{1}{k \Gamma_k(\alpha)} \int_x^b (t - x)^{\frac{\alpha}{k} - 1} f(t) \, dt, \quad b > x,
\end{equation}

where $\Gamma_k(\cdot)$ stands for the $k$-gamma function. For $k = 1$, the $k$-fractional integrals yield Riemann–Liouville integrals. For $\alpha = k = 1$, the $k$-fractional integrals yield classical integrals. For more details, see [9, 12, 15, 19].

It is remarkable that Sarikaya et al. in [26], first give the following interesting integral inequalities of Hermite–Hadamard type involving Riemann–Liouville fractional integrals.

**Theorem 2.** Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1 [a, b]$. If $f$ is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

\begin{equation}
f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2},
\end{equation}

with $\alpha > 0$.

Budak et al. in [1], prove the following Hermite-Hadamard type inequalities for $F$-convex functions via fractional integrals:

**Theorem 3.** Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on $I^\circ$, $a, b \in I^\circ$, $a < b$. If $f$ is $F$-convex on $[a, b]$ for some $F \in \mathcal{F}$, then we have

\begin{equation}
F \left( f \left( \frac{a + b}{2} \right) , \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha} J_{a+}^\alpha f(b) , \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha} J_{b-}^\alpha f(a) , \frac{1}{2} \right) \\
+ \int_0^1 L_{w(t)} \, dt \leq 0,
\end{equation}

and

\begin{equation}
T_{F,w} \left( \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] , f(a) + f(b) , f(a) + f(b) \right) \\
+ \int_0^1 L_{w(t)} \, dt \leq 0,
\end{equation}

where $w(t) = \alpha t^{\alpha - 1}$.

For other papers involving $F$-convex functions, see [1]-[4], [16, 27].

Now we summarize the generalized fractional integrals defined by Sarikaya and Erdal in [22].

Let’s define a function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

\begin{equation}
\int_0^1 \frac{\varphi(t)}{t} \, dt < +\infty,
\end{equation}
\[ \frac{1}{A_1} \leq \frac{\varphi(v)}{\varphi(u)} \leq A_1 \text{ for } \frac{1}{2} \leq \frac{u}{v} \leq 2, \]

\[ \frac{\varphi(u)}{u^2} \leq A_2 \frac{\varphi(v)}{v^2} \text{ for } v \leq u, \]

\[ \left| \frac{\varphi(u)}{u^2} - \frac{\varphi(v)}{v^2} \right| \leq A_3 |u-v| \frac{\varphi(u)}{u^3} \text{ for } \frac{1}{2} \leq \frac{u}{v} \leq 2, \]

where \( A_1, A_2, A_3 > 0 \) are independent of \( u, v > 0 \). If \( \varphi(u)u^\alpha \) is increasing for some \( \alpha \geq 0 \) and \( \frac{\varphi(u)}{u^\beta} \) is decreasing for some \( \beta \geq 0 \), then \( \varphi \) satisfies the above conditions.

The following left-sided and right-sided generalized fractional integral operators are defined respectively, as follows:

\[ a^+I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t) dt, \quad x > a, \]

\[ b^-I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t) dt, \quad x < b. \]

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integral, \( k \)-Riemann–Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc.

Sarikaya and Ertuğral in [22], establish the following Hermite–Hadamard inequality and lemmas for the generalized fractional integral operators:

**Theorem 4.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\) with \( a < b \), then the following inequalities for fractional integral operators hold:

\[ f \left( \frac{a+b}{2} \right) \leq \frac{1}{2\Psi(1)} [a^+I_\varphi f(b) + b^-I_\varphi f(a)] \leq \frac{f(a) + f(b)}{2}, \]

where the mapping \( \Lambda : [0, 1] \to \mathbb{R} \) is defined by

\[ \Psi(x) = \int_0^x \varphi \left( \frac{(b-a)t}{t} \right) dt. \]

Budak et al. prove the following Hermite Hadamard type inequalities for \( F \)-convex functions.

**Theorem 5 ([4]).** Let \( I \subseteq \mathbb{R} \) be an interval, \( f : I^\circ \subseteq \mathbb{R} \to \mathbb{R} \) be a mapping on \( I^\circ \), \( a, b \in I^\circ \), \( a < b \). If \( f \) is \( F \)-convex on \([a, b]\) for some \( F \in \mathcal{F} \), then we have

\[ F \left( f \left( \frac{a+b}{2} \right), \frac{1}{\Psi(1)} a^+I_\varphi f(b), \frac{1}{\Psi(1)} b^-I_\varphi f(a), \frac{1}{2} \right) + \int_0^1 Lw(t) dt \leq 0, \]
and

\[ T_{F,w} \left( \frac{1}{\Psi(1)} \left[ a + I_{\varphi} f(b) + b - I_{\varphi} f(a) \right], f(a) + f(b), f(a) + f(b) \right) \]

\[ + \int_0^1 L_{w(t)} \, dt \leq 0, \]

where \( w(t) = \frac{\varphi((b-a)t)}{t\Psi(1)}. \)

Motivated by the above literatures, the main objective of this article is to establish some new Hermite–Hadamard type inequalities via generalized fractional integrals utilizing \( F \)-convex functions. Some special cases of our main results recaptured the well–known earlier works. At the end, a briefly conclusion will be given as well.

2. Main results

In this section, we establish some inequalities of Hermite–Hadamard type including generalized fractional integrals via \( F \)-convex functions.

**Theorem 6.** Let \( I \subseteq \mathbb{R} \) be an interval, \( f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R} \) be a mapping on \( I^{\circ} \), \( a, b \in I^{\circ} \), \( a < b \) and let \( F \) be linear with respect to the first three variables. If \( f \) is \( F \)-convex on \([a,b]\) for some \( F \in \mathcal{F} \), then we have

\[
F \left( f \left( \frac{a + b}{2} \right), \frac{1}{\Lambda(1)} \left( \frac{e^{a+b}}{2} + I_{\varphi} f(b) \right), \frac{1}{\Lambda(1)} \left( \frac{e^{a+b}}{2} - I_{\varphi} f(a) \right), \frac{1}{2} \right) 
\]

\[ + \int_0^1 L_{w(t)} \, dt \leq 0, \]

and

\[
T_{F,w} \left( \frac{1}{\Lambda(1)} \left[ \left( \frac{e^{a+b}}{2} + I_{\varphi} f(b) \right) + \left( \frac{e^{a+b}}{2} - I_{\varphi} f(a) \right) \right], f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{w(t)} \, dt \leq 0, \]

where \( w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}. \) and the function \( \Lambda : [0, 1] \to \mathbb{R} \) is defined by

\[
\Lambda(x) = \int_0^x \frac{\varphi \left( \frac{b-a}{2} t \right)}{t} \, dt.
\]

**Kanıt.** Since \( f \) is \( F \)-convex, we have

\[
F \left( f \left( \frac{x + y}{2} \right), f(x), f(y), \frac{1}{2} \right) \leq 0, \quad \forall x, y \in [a, b].
\]
For
\[ x = \frac{t}{2}a + \left(\frac{2-t}{2}\right)b \quad \text{and} \quad y = \left(\frac{2-t}{2}\right)a + \frac{t}{2}b, \]
we have
\[ F\left(f\left(\frac{a+b}{2}\right), f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right), f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right), \frac{1}{2}\right) \leq 0. \]
for all \( t \in [0,1] \). Multiplying this inequality by \( w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)} \) and using axiom (A3), we get
\[ F\left(\varphi\left(\frac{(b-a)t}{t\Lambda(1)}\right)f\left(\frac{a+b}{2}\right), \varphi\left(\frac{(b-a)t}{t\Lambda(1)}\right)f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right), \varphi\left(\frac{(b-a)t}{t\Lambda(1)}\right)f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right), \frac{1}{2}\right) + Lw(t) \leq 0 \]
for all \( t \in (0,1) \). Integrating over \((0,1)\) with respect to the variable \( t \) and using axiom (A1), we obtain
\[ F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)} \int_0^1 \varphi\left(\frac{(b-a)t}{t}\right) dt, \frac{1}{\Lambda(1)} \int_0^1 \varphi\left(\frac{(b-a)t}{t}\right) f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) dt, \frac{1}{\Lambda(1)} \int_0^1 \varphi\left(\frac{(b-a)t}{t}\right) f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right) dt, \frac{1}{2}\right) + \int_0^1 Lw(t) dt \leq 0. \]
Using the facts that
\[ \int_0^1 \varphi\left(\frac{(b-a)t}{t}\right) f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) dt = \int_{\frac{a+b}{2}}^b \varphi(\frac{b-x}{b-x}) f(x) dx = \left(\frac{a+b}{2}\right) + I_{\varphi} f(b) \]
and
\[ \int_0^1 \varphi\left(\frac{(b-a)t}{t}\right) f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right) dt = \int_{\frac{a+b}{2}}^a \varphi(\frac{x-a}{x-a}) f(x) dx = \left(\frac{a+b}{2}\right) - I_{\varphi} f(a), \]
we obtain
\[ F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)} \left(\frac{a+b}{2}\right) + I_{\varphi} f(b), \frac{1}{\Lambda(1)} \left(\frac{a+b}{2}\right) - I_{\varphi} f(a), \frac{1}{2}\right) + \int_0^1 Lw(t) dt \leq 0, \]
which gives (18).

On the other hand, since $f$ is $F$–convex, we have

$$F \left( f \left( \frac{t}{2}a + \left( \frac{2-t}{2} \right)b \right), f(a), f(b), t \right) \leq 0, \quad \forall t \in [0,1],$$

and

$$F \left( f \left( \left( \frac{2-t}{2} \right) a + \frac{t}{2} b \right), f(a), f(b), t \right) \leq 0, \quad \forall t \in [0,1].$$

Using the linearity of $F$, we get

$$F \left( f \left( \frac{t}{2}a + \left( \frac{2-t}{2} \right) b \right) + f \left( \left( \frac{2-t}{2} \right) a + \frac{t}{2} b \right), f(a) + f(b), f(a) + f(b), t \right) \leq 0,$$

for all $t \in [0,1]$. Applying the axiom (A3) for $w(t) = \frac{\varphi \left( \frac{(b-a)t}{2} \right)}{t \Lambda(1)}$, we obtain

$$F \left( \frac{\varphi \left( \frac{(b-a)t}{2} \right)}{t \Lambda(1)} \left[ f \left( \frac{t}{2}a + \left( \frac{2-t}{2} \right) b \right) + f \left( \left( \frac{2-t}{2} \right) a + \frac{t}{2} b \right) \right], \varphi \left( \frac{(b-a)t}{2} \right) \frac{t}{t \Lambda(1)} [f(a) + f(b)] , \frac{\varphi \left( \frac{(b-a)t}{2} \right)}{t \Lambda(1)} f(a) + f(b), t \right) + L_{w(t)} \leq 0,$$

for all $t \in (0,1)$. Integrating over $(0,1)$ and using axiom (A2), we have

$$T_{F,w} \left( \int_0^1 \frac{\varphi \left( \frac{(b-a)t}{2} \right)}{t \Lambda(1)} \left[ f \left( \frac{t}{2}a + \left( \frac{2-t}{2} \right) b \right) + f \left( \left( \frac{2-t}{2} \right) a + \frac{t}{2} b \right) \right] dt, \right.$$

$$f(a) + f(b), f(a) + f(b) + \int_0^1 L_{w(t)} dt \leq 0,$$

that is

$$T_{F,w} \left( \frac{1}{\Lambda(1)} \left[ \left( \frac{a+b}{2} \right)^{I_\varphi f(b)} f(b) + \left( \frac{a+b}{2} \right)^{I_\varphi f(a)} f(a) \right], f(a) + f(b), f(a) + f(b) \right)$$

$$+ \int_0^1 L_{w(t)} dt \leq 0.$$

The proof of Theorem 6 is completed. $\square$
**Remark 2.** If we choose $\varphi(t) = t$ in Theorem 6, then we have the following inequalities

$$
F \left( f \left( \frac{a+b}{2} \right), \frac{2^b}{b-a} \int_a^b f(t) dt, \frac{2^a}{b-a} \int_a^{\frac{a+b}{2}} f(t) dt, \frac{1}{2} \right) 
+ \int_0^1 L_{w(t)} dt \leq 0,
$$

and

$$
T_{F,w} \left( \frac{2^b}{b-a} \int_a^b f(t) dt, f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{w(t)} dt \leq 0,
$$

where $w(t) = 1$.

**Remark 3.** If we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 6, then we have the following inequalities for Riemann-Liouville fractional integrals

$$
F \left( f \left( \frac{a+b}{2} \right), \frac{2^\alpha \Gamma(\alpha + 1)}{(b-a)^\alpha} J_{\frac{a+b}{2}}^\alpha f(b), \frac{2^\alpha \Gamma(\alpha + 1)}{(b-a)^\alpha} J_{\frac{a+b}{2}}^- f(a), \frac{1}{2} \right) 
+ \int_0^1 L_{w(t)} dt \leq 0,
$$

and

$$
T_{F,w} \left( \frac{2^\alpha \Gamma(\alpha + 1)}{(b-a)^\alpha} \left[ J_{\frac{a+b}{2}}^\alpha f(b) + J_{\frac{a+b}{2}}^- f(a) \right], f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{w(t)} dt \leq 0,
$$

where $w(t) = \alpha t^{\alpha-1}$ which is given by Budak et al. in [5].

**Corollary 1.** If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)} k$ in Theorem 6, then we have the following inequalities for $k$–Riemann–Liouville fractional integrals

$$
F \left( f \left( \frac{a+b}{2} \right), \frac{2^\alpha \Gamma_k(\alpha + k)}{(b-a)^\alpha} I_{\frac{a+b}{2}}^\alpha f(b), \frac{2^\alpha \Gamma_k(\alpha + k)}{(b-a)^\alpha} I_{\frac{a+b}{2}}^- f(a), \frac{1}{2} \right) 
+ \int_0^1 L_{w(t)} dt \leq 0,
$$

and

$$
T_{F,w} \left( \frac{2^\alpha \Gamma_k(\alpha + k)}{(b-a)^\alpha} \left[ I_{\frac{a+b}{2}}^\alpha f(b) + I_{\frac{a+b}{2}}^- f(a) \right], f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{w(t)} dt \leq 0,
$$

where $w(t) = \alpha t^{\alpha-1}$.
New inequalities for \( F \)-convex functions

\[
f(a) + f(b), f(a) + f(b) + \int_0^1 L_w(t) dt \leq 0,
\]

where \( w(t) = \alpha t^\frac{n}{k} \).

**Theorem 7.** Let \( I \subseteq \mathbb{R} \) be an interval, \( f : I^o \subseteq \mathbb{R} \to \mathbb{R} \) be a mapping on \( I^o \), \( a, b \in I^o \), \( a < b \) and let \( F \) be linear with respect to the first three variables. If \( f \) is \( F \)-convex on \([a, b]\) for some \( F \in \mathcal{F} \), then we have

\[
F\left( f\left( \frac{a+b}{2} \right), \frac{1}{\Lambda(1)} b - I_\varphi f\left( \frac{a+b}{2} \right) \right), \quad (22)
\]

\[
\frac{1}{\Lambda(1)} a + I_\varphi f\left( \frac{a+b}{2} \right), \frac{1}{2} \right) + \int_0^1 L_w(t) dt \leq 0,
\]

and

\[
T_{F,w} \left( \frac{1}{\Lambda(1)} \left[ a + I_\varphi f\left( \frac{a+b}{2} \right) + b - I_\varphi f\left( \frac{a+b}{2} \right) \right] \right), \quad (23)
\]

\[
f(a) + f(b), f(a) + f(b) + \int_0^1 L_w(t) dt \leq 0,
\]

where \( w(t) = \varphi\left( \frac{b-a}{2} \right) \).

**Kanıt.** Since \( f \) is \( F \)-convex, we have

\[
F\left( f\left( \frac{x+y}{2} \right), f(x), f(y), \frac{1}{2} \right) \leq 0, \quad \forall x, y \in [a, b].
\]

For

\[
x = \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \quad \text{and} \quad y = \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) b,
\]

we have

\[
F\left( f\left( \frac{a+b}{2} \right), f\left( \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \right) \right), \quad f\left( \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) b \right), \frac{1}{2} \right) \leq 0,
\]

for all \( t \in [0,1] \). Multiplying this inequality by \( w(t) = \varphi\left( \frac{b-a}{2} \right) \) and using axiom (A3), we get

\[
F\left( \varphi\left( \frac{b-a}{2} \right) t \right) f\left( \frac{a+b}{2} \right), \varphi\left( \frac{b-a}{2} \right) t f\left( \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \right), \quad (24)
\]

\[
\varphi\left( \frac{b-a}{2} \right) t f\left( \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) b \right), \frac{1}{2} \right) + L_w(t) \leq 0,
\]
for all $t \in (0,1)$. Integrating over $(0,1)$ with respect to the variable $t$ and using axiom (A1), we obtain

$$F\left(\frac{f \left(\frac{a+b}{2}\right)}{\Lambda(1)} \int_0^1 \frac{\varphi \left(\frac{(b-a)}{2} t\right)}{t} \, dt\right),$$

$$\frac{1}{\Lambda(1)} \int_0^1 \frac{\varphi \left(\frac{(b-a)}{2} t\right)}{t} f \left(\left(\frac{1-t}{2}\right) a + \left(\frac{1+t}{2}\right) b\right) \, dt,$$

$$\frac{1}{\Lambda(1)} \int_0^1 \frac{\varphi \left(\frac{(b-a)}{2} t\right)}{t} f \left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right) \, dt, \frac{1}{2}\right)$$

$$+ \int_0^1 L_{w(t)} \, dt \leq 0.$$  

Using the facts that

$$\int_0^1 \varphi \left(\frac{(b-a)}{2} t\right) f \left(\left(\frac{1-t}{2}\right) a + \left(\frac{1+t}{2}\right) b\right) \, dt$$

$$= \int_{\frac{a+b}{2}}^b \varphi \left(\frac{x-a+b}{2}\right) \frac{x-a+b}{2} f(x) \, dx$$

$$= b - I_{\varphi} f \left(\frac{a+b}{2}\right),$$

and

$$\int_0^1 \varphi \left(\frac{(b-a)}{2} t\right) f \left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right) \, dt$$

$$= \int_{\frac{a+b}{2}} a \varphi \left(\frac{a+b}{2} - x\right) \frac{a+b}{2} - x f(x) \, dx$$

$$= a + I_{\varphi} f \left(\frac{a+b}{2}\right),$$

we obtain

$$F\left(f \left(\frac{a+b}{2}\right), \varfrac{1}{\Lambda(1)} b - I_{\varphi} f \left(\frac{a+b}{2}\right), \varfrac{1}{\Lambda(1)} a + I_{\varphi} f \left(\frac{a+b}{2}\right), \frac{1}{2}\right)$$

$$+ \int_0^1 L_{w(t)} \, dt \leq 0,$$  

which gives (22).

On the other hand, since $f$ is $F$–convex, we have

$$F\left(f \left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right), f(a), f(b), t\right) \leq 0, \quad \forall t \in [0,1],$$
and
\[ F \left( f \left( \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \right), f(a), f(b), t \right) \leq 0, \quad \forall t \in [0, 1]. \]

Using the linearity of \( F \), we get
\[
F \left( f \left( \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) b \right) + f \left( \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \right) \right),
\]
\[ f(a) + f(b), f(a) + f(b), t \] \leq 0, \quad \forall t \in [0, 1].

Applying the axiom (A3) for \( w(t) = \varphi \left( \frac{(b-a)t}{2} \right) \), we obtain
\[
F \left( \frac{\varphi \left( \frac{(b-a)t}{2} \right)}{t \Lambda(1)} \times \left[ f \left( \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) b \right) + f \left( \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \right) \right], \varphi \left( \frac{(b-a)t}{2} \right) \right] + \frac{L w(t)}{t \Lambda(1)} + L w(t) \leq 0,
\]
for all \( t \in (0, 1). \) Integrating over \((0, 1)\) and using axiom (A2), we have
\[
T_{F,w} \left( \int_{0}^{1} \frac{\varphi \left( \frac{(b-a)t}{2} \right)}{t \Lambda(1)} \times \left[ f \left( \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) b \right) + f \left( \left( \frac{1-t}{2} \right) a + \left( \frac{1+t}{2} \right) b \right) \right] dt, \right.
\]
\[ f(a) + f(b), f(a) + f(b), t \right] + \int_{0}^{1} L w(t) dt \leq 0,
\]
that is
\[
T_{F,w} \left( \frac{1}{\Lambda(1)} \left[ a + I \varphi f \left( \frac{a+b}{2} \right) + b - I \varphi f \left( \frac{a+b}{2} \right) \right], \right.
\]
\[ f(a) + f(b), f(a) + f(b), t \right] \leq \int_{0}^{1} L w(t) dt \leq 0.
\]

The proof of Theorem 7 is completed. \( \square \)

**Remark 4.** If we take \( \varphi(t) = t \) in Theorem 7, then the inequalities (22) and (23) reduce to the inequalities (20) and (21)
Remark 5. If we take \( \varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)} \) in Theorem 7, then we have the following inequalities for Riemann-Liouville fractional integrals
\[
F\left( f\left( \frac{a+b}{2} \right), \frac{2^\alpha \Gamma(\alpha + 1)}{(b-a)^\alpha} J_{b^-}^\alpha f\left( \frac{a+b}{2} \right), \frac{2^\alpha \Gamma(\alpha + 1)}{(b-a)^\alpha} J_{a^+}^\alpha f\left( \frac{a+b}{2} \right), \frac{1}{2} \right) + \int_0^1 L_w(t) \, dt \leq 0,
\]
and
\[
T_{F,w}\left( \frac{2^\alpha \Gamma(\alpha + 1)}{(b-a)^\alpha} \left[ J_{b^-}^\alpha f\left( \frac{a+b}{2} \right) + J_{a^+}^\alpha f\left( \frac{a+b}{2} \right) \right] \right),
\]
\[
f(a) + f(b), f(a) + f(b) + \int_0^1 L_w(t) \, dt \leq 0,
\]
where \( w(t) = \alpha t^{\alpha-1} \) which is given by Budak et al. in [5].

Corollary 2. If we take \( \varphi(t) = \frac{t^\alpha}{\Gamma_k(\alpha)} \) in Theorem 7, then we have the following inequalities for \( k \)-Riemann-Liouville fractional integrals:
\[
F\left( f\left( \frac{a+b}{2} \right), \frac{2^\alpha \Gamma_k(\alpha + k)}{(b-a)^\alpha_k} I_{b^-}^\alpha, k f\left( \frac{a+b}{2} \right), \frac{2^\alpha \Gamma_k(\alpha + k)}{(b-a)^\alpha_k} I_{a^+}^\alpha, k f\left( \frac{a+b}{2} \right), \frac{1}{2} \right) + \int_0^1 L_w(t) \, dt \leq 0,
\]
and
\[
T_{F,w}\left( \frac{2^\alpha \Gamma_k(\alpha + k)}{(b-a)^\alpha_k} \left[ I_{b^-}^\alpha f\left( \frac{a+b}{2} \right) + I_{a^+}^\alpha f\left( \frac{a+b}{2} \right) \right] \right),
\]
\[
f(a) + f(b), f(a) + f(b) + \int_0^1 L_w(t) \, dt \leq 0,
\]
where \( w(t) = \frac{\alpha}{k} t^{\frac{\alpha}{k}-1} \).

Remark 6. One can obtain several results for convexity, \( \varepsilon \)-convexity, \( h \)-convexity, etc by special choice of the function \( F \) in Theorems 6 and 7.

3. Conclusion

In the development of this work, using the definition of \( F \)-convex functions some new Hermite-Hadamard type inequalities via generalized fractional integrals have been deduced. We also give several results capturing Riemann-Liouville fractional integrals and \( k \)-Riemann-Liouville fractional integrals as special cases. The authors hope that these results will serve as a motivation for future work in this fascinating area.
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