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On the bi-periodic Padovan sequences

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Abstract. In this study, we define a new generalization of the Padovan numbers, which shall also be called the bi-periodic Padovan sequence. Also, we consider a generalized bi-periodic Padovan matrix sequence. Finally, we investigate the Binet formulas, generating functions, series and partial sum formulas for these sequences.

1. Introduction

Due to numerous applications of some integer sequences, such as Fibonacci, Lucas, Pell, Padovan, etc., many fields of science and art, many generalizations have been made about them in the last century. Their beauty and ubiquity continue to amaze the mathematics community (for details see [16–18]). In this paper, our main focus is to define a new generalization for the Padovan sequence.

The bi-periodic Fibonacci numbers, also known as the generalized Fibonacci numbers, were first described by Edson and Yayenie [11] as

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even;} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \qquad n \geq 2,$$

with initial conditions $q_0 = 0$ and $q_1 = 1$, where a and b are any nonzero real numbers. In a similar way, the bi-periodic Lucas sequence was defined by Bilgici [3], and the bi-periodic Jacobsthal sequence was defined by Uygun and Owuso [26]. In [5], some new identities involving differences in products of generalized Fibonacci numbers are shown. Irmak et al. have presented various studies on periodic functions [1, 14, 15]. Various identities have been generalized by many researchers [2, 4, 6, 7, 19, 21–25, 27].

In the present paper, just as with the generalized Fibonacci sequence and others, we define a generalization for the Padovan sequence, which we call the bi-periodic Padovan sequence. We obtain its Binet-like formula and its generating function. Also, we define the matrix representation of the

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bi-periodic Padovan sequence and give various identities for the bi-periodic Padovan sequence.

The Padovan sequence $\{P_n\}_{n\geq 0}$ is defined by the third order recurrence

$$P_{n+3} = P_{n+1} + P_n \tag{1}$$

with the initial conditions $P_0 = 1$, $P_1 = 0$ and $P_2 = 1$. For relevance, we consider as $P_{-2} = P_{-1} = 0$. The first few values of this sequence are

$$P_0 = 1,\ \ P_1 = 0,\ \ P_2 = 1,\ \ P_3 = 1,\ \ P_4 = 1,$$
$$P_5 = 2,\ \ P_6 = 2,\ \ P_7 = 3,\ \ P_8 = 4,\ \ P_9 = 5.$$

The recurrence (1) involves the characteristic equation

$$x^3 - x - 1 = 0.$$

If its roots are denoted by α, β and γ then, the following equalities can be derived

$$\alpha + \beta + \gamma = 0, \quad \alpha\beta + \alpha\gamma + \beta\gamma = -1, \quad \alpha\beta\gamma = 1.$$

Moreover, the Binet-like formula for the Padovan sequence is

$$P_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n + \hat{\gamma}\gamma^n, \tag{2}$$

where

$$\hat{\alpha} = \frac{\beta\gamma + 1}{(\alpha - \beta)(\alpha - \gamma)}, \quad \hat{\beta} = \frac{\alpha\gamma + 1}{(\beta - \alpha)(\beta - \gamma)}, \quad \hat{\alpha} = \frac{\alpha\beta + 1}{(\gamma - \alpha)(\gamma - \beta)}.$$

More information is available in [8–10, 12, 13, 20] for the Padovan numbers.

2. Main results

In this section, a new generalization of the Padovan numbers is defined, which shall also be called the bi-periodic Padovan sequence. Also, the Binet formulas, generating functions, series, and partial sum formulas for these sequences are investigated.

Definition 1. The bi-periodic Padovan sequences indicated by $\{p_n\}_{n\geq 0}$ is defined by

$$p_n = \begin{cases} ap_{n-2} + p_{n-3}, & \text{if } n \text{ is even;} \\ bp_{n-2} + p_{n-3}, & \text{if } n \text{ is odd.} \end{cases} \qquad n \geq 2,$$

with the initial conditions $p_0 = 1$, $p_1 = 0$ and $p_2 = a$, where a and b are nonzero real numbers.

The first few elements of the bi-periodic Padovan sequences are

$$p_0 = 1,\ \ p_1 = 0,\ \ p_2 = a,\ \ p_3 = 1,\ \ p_4 = a^2,$$
$$p_5 = a + b,\ \ p_6 = a^3 + 1,\ \ p_7 = b^2 + ab + a^2.$$

From the definition above, we obtain a nonlinear cubic equation for the bi-periodic Padovan sequences by

$$x^3 - abx - ab = 0. \tag{3}$$

By λ, μ and δ we denote the roots of the equation (3). Hence, the following relations are valid:

$$\lambda + \mu + \delta = 0, \quad \lambda\mu\delta = ab, \quad \lambda\mu + \lambda\delta + \mu\delta = -ab.$$

Theorem 1. *The Binet-like formula of the bi-periodic Padovan numbers is*

$$p_n = \begin{cases} x\lambda^n + y\mu^n + z\delta^n, & \text{if } n \text{ is even;} \\ u\lambda^n + v\mu^n + w\delta^n, & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 0, \tag{4}$$

with

$$x = \frac{(\mu - a)(\delta - a)}{(\lambda - \mu)(\lambda - \delta)}, \quad y = \frac{(\lambda - a)(\delta - a)}{(\mu - \lambda)(\mu - \delta)}, \quad z = \frac{(\lambda - a)(\mu - a)}{(\delta - \lambda)(\delta - \mu)},$$

and

$$u = \frac{a + b - \mu - \delta}{(\lambda - \mu)(\lambda - \delta)}, \quad v = \frac{a + b - \lambda - \delta}{(\mu - \lambda)(\mu - \delta)}, \quad w = \frac{a + b - \lambda - \mu}{(\delta - \lambda)(\delta - \mu)}.$$

Proof. With the characteristic equation (3) and its roots λ, μ and δ, assume the general solution of the difference equation above is

$$p_n = \begin{cases} x\lambda^n + y\mu^n + z\delta^n, & n = 2k; \\ u\lambda^n + v\mu^n + w\delta^n, & n = 2k + 1, \end{cases} \quad k \in \mathbb{N}.$$

Now, we determine the coefficients x, y, z, u, v and w. By the initial conditions, we have

$$\begin{aligned} p_0 &= x + y + z = 1, \\ p_2 &= x\lambda^2 + y\mu^2 + z\delta^2 = a, \\ p_4 &= x\lambda^2 + y\mu^2 + z\delta^2 = a^2, \end{aligned}$$

and

$$\begin{aligned} p_1 &= u\lambda + v\mu + w\delta = 0, \\ p_3 &= u\lambda^3 + v\mu^3 + w\delta^3 = 1, \\ p_5 &= u\lambda^5 + v\mu^5 + w\delta^5 = a + b. \end{aligned}$$

We can obtain that

$$x = \frac{(\mu - a)(\delta - a)}{(\lambda - \mu)(\lambda - \delta)}, \quad y = \frac{(\lambda - a)(\delta - a)}{(\mu - \lambda)(\mu - \delta)}, \quad z = \frac{(\lambda - a)(\mu - a)}{(\delta - \lambda)(\delta - \mu)}$$

and

$$u = \frac{a + b - \mu - \delta}{(\lambda - \mu)(\lambda - \delta)}, \quad v = \frac{a + b - \lambda - \delta}{(\mu - \lambda)(\mu - \delta)}, \quad w = \frac{a + b - \lambda - \mu}{(\delta - \lambda)(\delta - \mu)}. \qquad \square$$

Proposition 1. *The bi-periodic Padovan sequence* $\{p_n\}_{n\geq 0}$ *convinces with the following properties:*

$$p_{2k+1} = (a+b)p_{2k-1} - abp_{2k-3} + p_{2k-5}, \tag{5}$$

$$p_{2k} = (a+b)p_{2k-2} - abp_{2k-4} + p_{2k-6}. \tag{6}$$

Proof. Using Definition 1, we get

$$\begin{aligned} p_{2k+1} &= bp_{2k-1} + p_{2k-2} = bp_{2k-1} + ap_{2k-4} + p_{2k-5} \\ &= bp_{2k-1} + a(p_{2k-1} - bp_{2k-3}) + p_{2k-5} \\ &= bp_{2k-1} + ap_{2k-1} - abp_{2k-3} + p_{2k-5} \\ &= (a+b)p_{2k-1} - abp_{2k-3} + p_{2k-5}. \end{aligned}$$

Similarly, by Definition 1, we have

$$\begin{aligned} p_{2k} &= ap_{2k-2} + p_{2k-3} = ap_{2k-2} + bp_{2k-5} + p_{2k-6} \\ &= ap_{2k-2} + b(p_{2k-2} - ap_{2k-4}) + p_{2k-6} \\ &= ap_{2k-2} + bp_{2k-2} - abp_{2k-4} + p_{2k-6} \\ &= (a+b)p_{2k-2} - abp_{2k-4} + p_{2k-6}. \end{aligned}$$

□

Theorem 2. *The generating function for the bi-periodic Padovan sequences is*

$$G_p(x) = \frac{-x^3 + bx^2 - 1}{x^6 - abx^4 + (a+b)x^2 - 1}. \tag{7}$$

Proof. Let us assume that $G_p(x) = \sum_{k=0}^{\infty} p_k x^k$ is the generation function of the bi-periodic Padovan sequences. Thus, we have

$$G_p(x) = \sum_{k=0}^{\infty} p_{2k}x^{2k} + \sum_{k=0}^{\infty} p_{2k+1}x^{2k+1}.$$

Let $G_p^a(x) = \sum_{k=0}^{\infty} p_{2k}x^{2k}$ and $G_p^b(x) = \sum_{k=0}^{\infty} p_{2k+1}x^{2k+1}$. By multiplying $G_p^a(x)$ through by -1, $-abx^4$, $(a+b)x^2$ and x^6, respectively, we get

$$-G_p^a(x) = -p_0 - p_2x^2 - p_4x^4 - \sum_{k=3}^{\infty} p_{2k}x^{2k},$$

$$-abx^4G_p^a(x) = -abp_0x^4 - ab\sum_{k=3}^{\infty} p_{2k-4}x^{2k},$$

$$(a+b)x^2G_p^a(x) = (a+b)p_0x^2 + (a+b)p_2x^4 + (a+b)\sum_{k=3}^{\infty} p_{2k-2}x^{2k},$$

$$x^6G_p^a(x) = \sum_{k=3}^{\infty} p_{2k-6}x^{2k}.$$

Using (6), we have

$$G_p^a(x) = \frac{bx^2 - 1}{x^6 - abx^4 + (a+b)x^2 - 1}.$$

In a similar way, we obtain

$$G_p^b(x) = \frac{-x^3}{x^6 - abx^4 + (a+b)x^2 - 1}.$$

By summing the two parts side by side we obtain the desired result:

$$G_p(x) = \frac{-x^3 + bx^2 - 1}{x^6 - abx^4 + (a+b)x^2 - 1}.$$

Thus, the proof is completed. □

Theorem 3. *The series for the bi-periodic Padovan sequences is*

$$\sum_{k=0}^{\infty} \frac{p_k}{x^k} = \frac{x^6 - bx^4 + x^3}{x^6 - (a+b)x^4 + abx^2 - 1}.$$

Proof. It is proven by substituting $\frac{1}{x}$ instead of x in the equality (7). □

Theorem 4. *The exponential generating function for the bi-periodic Padovan sequences is*

$$\sum_{n=0}^{\infty} p_n \frac{t^n}{n!} = \begin{cases} xe^{\lambda t} + ye^{\mu t} + ze^{\delta t}, & \text{if } n \text{ is even;} \\ ue^{\lambda t} + ve^{\mu t} + we^{\delta t}, & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 0.$$

Proof. For $n \geq 0$ and using equality (4), we have

$$\sum_{n=0}^{\infty} p_n \frac{t^n}{n!} = \begin{cases} x \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} + y \sum_{n=0}^{\infty} \frac{(\mu t)^n}{n!} + z \sum_{n=0}^{\infty} \frac{(\delta t)^n}{n!}, & \text{if } n \text{ is even;} \\ u \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} + v \sum_{n=0}^{\infty} \frac{(\mu t)^n}{n!} + w \sum_{n=0}^{\infty} \frac{(\delta t)^n}{n!}, & \text{if } n \text{ is odd,} \end{cases}$$

$$= \begin{cases} xe^{\lambda t} + ye^{\mu t} + ze^{\delta t}, & \text{if } n \text{ is even;} \\ ue^{\lambda t} + ve^{\mu t} + we^{\delta t}, & \text{if } n \text{ is odd.} \end{cases}$$

□

Theorem 5. *The sum of the first n terms of the sequence $\{p_n\}$ is*

$$\sum_{k=0}^{n} p_k = \frac{p_{n-3} + p_{n-2} + p_{n+1} + p_{n+2} + (1-ab)(p_n + p_{n-1}) + b - 2}{a + b - ab}.$$

Proof. Let

$$\sum_{k=0}^{n} p_k = \sum_{k=0}^{\frac{n}{2}} p_{2k} + \sum_{k=0}^{\frac{n}{2}-1} p_{2k+1}.$$

The even part of the above sums is solved as follows. We know that (6):

$$p_{2k} = (a+b)p_{2k-2} - abp_{2k-4} + p_{2k-6}.$$

So, we have

$$p_{2k} - p_{2k-6} = (a+b)p_{2k-2} - abp_{2k-4}.$$

Applying to the identity above, we deduce that

$$\begin{aligned} p_6 - p_0 &= (a+b)p_4 - abp_2, \\ p_8 - p_2 &= (a+b)p_6 - abp_4, \\ p_{10} - p_4 &= (a+b)p_8 - abp_6, \\ &\vdots \\ p_n - p_{n-6} &= (a+b)p_{n-2} - abp_{n-4}. \end{aligned}$$

If we sum of both of sides of the identities above, we obtain,

$$p_{n+2} + p_n + p_{n-2} - abp_n + b - 1 = (a+b-ab)\sum_{k=0}^{\frac{n}{2}} p_{2k}.$$

The odd part is solved in a similar way and we get

$$p_{n+1} + p_{n-1} + p_{n-3} - abp_{n-1} - 1 = (a+b-ab)\sum_{k=0}^{\frac{n}{2}-1} p_{2k+1}.$$

Therefore, we obtain

$$\begin{aligned} \sum_{k=0}^{n} p_k &= \sum_{k=0}^{\frac{n}{2}} p_{2k} + \sum_{k=0}^{\frac{n}{2}-1} p_{2k+1} \\ &= \frac{p_{n-3} + p_{n-2} + p_{n+1} + p_{n+2} + (1-ab)(p_n + p_{n-1}) + b - 2}{a+b-ab}. \end{aligned}$$ □

3. The bi-periodic Padovan matrix sequence

In this section, the bi-periodic Padovan matrix sequence of the Padovan numbers is defined and the Binet formulas, generating functions, series, and partial sum formulas for these sequences are investigated.

Definition 2. Let $a, b \in \mathbb{R}-\{0\}$. The bi-periodic Padovan matrix sequences $\{Mp_n\}_{n\geq 0}$ is defined by

$$Mp_n = \begin{cases} aMp_{n-2} + Mp_{n-3}, & \text{if } n \text{ is even;} \\ bMp_{n-2} + Mp_{n-3}, & \text{if n is odd,} \end{cases} \qquad n \geq 3. \tag{8}$$

with initial conditions

$$Mp_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Mp_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a & 0 \end{bmatrix}, \quad Mp_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & a & 0 \\ 0 & 1 & a \end{bmatrix}.$$

Proposition 2. *Let $\{Mp_n\}_{n\geq 0}$ be the bi-periodic Padovan matrix sequence. The following relations are valid:*

$$Mp_{2k+1} = (a+b)\, Mp_{2k-1} - abMp_{2k-3} + Mp_{2k-5}, \tag{9}$$

$$Mp_{2k} = (a+b)\, Mp_{2k-2} - abMp_{2k-4} + Mp_{2k-6}. \tag{10}$$

Proof. Using (8), we have

$$\begin{aligned} Mp_{2k+1} &= bMp_{2k-1} + Mp_{2k-2} \\ &= bMp_{2k-1} + aMp_{2k-4} + Mp_{2k-5} \\ &= bMp_{2k-1} + a\left(Mp_{2k-1} - bMp_{2k-3}\right) + Mp_{2k-5} \\ &= bMp_{2k-1} + aMp_{2k-1} - abMp_{2k-3} + Mp_{2k-5} \\ &= (a+b)\, Mp_{2k-1} - abMp_{2k-3} + Mp_{2k-5}, \end{aligned}$$

and we write

$$\begin{aligned} p_{2k} &= aMp_{2k-2} + Mp_{2k-3} \\ &= aMp_{2k-2} + bMp_{2k-5} + Mp_{2k-6} \\ &= aMp_{2k-2} + b\left(Mp_{2k-2} - bMp_{2k-4}\right) + Mp_{2k-6} \\ &= aMp_{2k-2} + bMp_{2k-2} - abMp_{2k-4} + Mp_{2k-6} \\ &= (a+b)\, Mp_{2k-2} - abMp_{2k-4} + Mp_{2k-6}. \end{aligned}$$

□

Theorem 6. *The generating function for the bi-periodic Padovan matrix sequence $\{Mp_n\}_{n\geq 0}$ is*

$$G(x) = \begin{bmatrix} \frac{ax^5-abx^4-x^3+(a+b)x^2-1}{x^6-abx^4+(a+b)x^2-1} & \frac{x^4+ax^3-x}{x^6-abx^4+(a+b)x^2-1} & \frac{-x^5+bx^4-x^2}{x^6-abx^4+(a+b)x^2-1} \\ \frac{-x^5+bx^4-x^2}{x^6-abx^4+(a+b)x^2-1} & \frac{-x^3+bx^2-1}{x^6-abx^4+(a+b)x^2-1} & \frac{x^4+ax^3-x}{x^6-abx^4+(a+b)x^2-1} \\ \frac{x^4+ax^3-x}{x^6-abx^4+(a+b)x^2-1} & \frac{-x^5+(b-a)x^4+a^2x^3-x^2-ax}{x^6-abx^4+(a+b)x^2-1} & \frac{-x^3+bx^2-1}{x^6-abx^4+(a+b)x^2-1} \end{bmatrix}. \tag{11}$$

Proof. We know that

$$G(x) = \sum_{k=0}^{\infty} Mp_{2k}x^{2k} + \sum_{k=0}^{\infty} Mp_{2k+1}x^{2k+1}.$$

Let $G_a(x) = \sum_{k=0}^{\infty} Mp_{2k}x^{2k}$ and $G_b(x) = \sum_{k=0}^{\infty} Mp_{2k+1}x^{2k+1}$. Multiplying $G_a(x)$ by -1, $-abx^4$, $(a+b)\,x^2$ and x^6, respectively, we get

$$-G_a(x) = -Mp_0 - Mp_2x^2 - Mp_4x^4 - \sum_{k=3}^{\infty} Mp_{2k}x^{2k},$$

$$-abx^4G_a(x) = -abMp_0x^4 - ab\sum_{k=3}^{\infty} Mp_{2k-4}x^{2k},$$

$$(a+b)\,x^2G_a(x) = (a+b)\,Mp_0x^2 + (a+b)\,Mp_2x^4 + (a+b)\sum_{k=3}^{\infty} Mp_{2k-2}x^{2k},$$

$$x^6G_a(x)=\sum_{k=3}^{\infty}Mp_{2k-6}x^{2k}.$$

By the equalities above and using (9), we reach the following

$$G_a(x)=\begin{bmatrix}\frac{(a+b)\,x^2-abx^4-1}{x^6-abx^4+(a+b)\,x^2-1} & \frac{-x^4}{x^6-abx^4+(a+b)\,x^2-1} & \frac{-x^2+bx^4}{x^6-abx^4+(a+b)\,x^2-1}\\ \frac{bx^4-x^2}{x^6-abx^4+(a+b)\,x^2-1} & \frac{bx^2-1}{x^6-abx^4+(a+b)\,x^2-1} & \frac{x^4}{x^6-abx^4+(a+b)\,x^2-1}\\ \frac{x^4}{x^6-abx^4+(a+b)\,x^2-1} & \frac{(b-a)\,x^4-x^2}{x^6-abx^4+(a+b)\,x^2-1} & \frac{bx^2-1}{x^6-abx^4+(a+b)\,x^2-1}\end{bmatrix}.$$

Multiplying $G_b(x)$ by -1, $-abx^4$, $(a+b)\,x^2$ and x^6, respectively, we get

$$-G_b(x)=-Mp_1x-Mp_3x^3-Mp_5x^5-\sum_{k=3}^{\infty}Mp_{2k+1}x^{2k+1},$$

$$-abx^4G_b(x)=-abMp_1x^5-ab\sum_{k=3}^{\infty}Mp_{2k-3}x^{2k+1},$$

$$(a+b)\,x^2G_b(x)=(a+b)\,Mp_1x^3+(a+b)\,Mp_3x^5+(a+b)\sum_{k=3}^{\infty}Mp_{2k-1}x^{2k+1},$$

$$x^6G_b(x)=\sum_{k=3}^{\infty}Mp_{2k-5}x^{2k+1}.$$

By the equalities above and using (10), we reach the following

$$G_b(x)=\begin{bmatrix}\frac{ax^5-x^3}{x^6-abx^4+(a+b)\,x^2-1} & \frac{ax^3-x}{x^6-abx^4+(a+b)\,x^2-1} & \frac{-x^5}{x^6-abx^4+(a+b)\,x^2-1}\\ \frac{-x^5}{x^6-abx^4+(a+b)\,x^2-1} & \frac{-x^3}{x^6-abx^4+(a+b)\,x^2-1} & \frac{ax^3-x}{x^6-abx^4+(a+b)\,x^2-1}\\ \frac{ax^3-x}{x^6-abx^4+(a+b)\,x^2-1} & \frac{a^2x^3-x^5-ax}{x^6-abx^4+(a+b)\,x^2-1} & \frac{-x^3}{x^6-abx^4+(a+b)\,x^2-1}\end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned}G(x)&=G_a(x)+G_b(x)\\ &=\begin{bmatrix}\frac{ax^5-abx^4-x^3+(a+b)x^2-1}{x^6-abx^4+(a+b)x^2-1} & \frac{-x^4+ax^3-x}{x^6-abx^4+(a+b)x^2-1} & \frac{-x^5+bx^4-x^2}{x^6-abx^4+(a+b)x^2-1}\\ \frac{-x^5+bx^4-x^2}{x^6-abx^4+(a+b)x^2-1} & \frac{-x^3+bx^2-1}{x^6-abx^4+(a+b)x^2-1} & \frac{-x^4+ax^3-x}{x^6-abx^4+(a+b)x^2-1}\\ \frac{-x^4+ax^3-x}{x^6-abx^4+(a+b)x^2-1} & \frac{-x^5+(b-a)x^4+a^2x^3-x^2-ax}{x^6-abx^4+(a+b)x^2-1} & \frac{-x^3+bx^2-1}{x^6-abx^4+(a+b)x^2-1}\end{bmatrix}.\end{aligned}$$ □

Theorem 7. *The series of the bi-periodic Padovan matrix sequence* $\{Mp_n\}_{n\geq 0}$ *is*

$$S(x)=\begin{bmatrix}\frac{x^6-(a+b)\,x^4+x^3+abx^2-ax}{x^6-(a+b)\,x^4+abx^2-1} & \frac{x^5-ax^3+x^2}{x^6-(a+b)\,x^4+abx^2-1} & \frac{x^4-bx^2+x}{x^6-(a+b)\,x^4+abx^2-1}\\ \frac{x^4-bx^2+x}{x^6-(a+b)\,x^4+abx^2-1} & \frac{x^6-bx^4+x^3}{x^6-(a+b)\,x^4+abx^2-1} & \frac{x^5-ax^3+x^2}{x^6-(a+b)\,x^4+abx^2-1}\\ \frac{x^5-ax^3+x^2}{x^6-(a+b)\,x^4+abx^2-1} & \frac{ax^5+x^4-a^2x^3+(a-b)\,x^2+x}{x^6-(a+b)\,x^4+abx^2-1} & \frac{x^6-bx^4+x^3}{x^6-(a+b)\,x^4+abx^2-1}\end{bmatrix}.$$

Proof. It is proven by substituting $\frac{1}{x}$ instead of x in the equality (11). □

Theorem 8. *The sum of the first n terms of the matrix sequence* $\{Mp_n\}$ *is*

$$\sum_{k=0}^{n} Mp_k = \frac{Mp_{n+2} + Mp_{n+1} + Mp_{n-2} + Mp_{n-3} + (1-ab)\left(Mp_{n-1} + Mp_n\right)}{a+b-ab}$$
$$+ \begin{bmatrix} \frac{2a+b-ab-1}{a+b-ab} & \frac{a-2}{a+b-ab} & \frac{b-2}{a+b-ab} \\ \frac{b-2}{a+b-ab} & \frac{b-2}{a+b-ab} & \frac{a-2}{a+b-ab} \\ \frac{a-2}{a+b-ab} & \frac{a^2+b-2a-2}{a+b-ab} & \frac{b-2}{a+b-ab} \end{bmatrix}.$$

Proof. Let

$$\sum_{k=0}^{n} Mp_k = \sum_{k=0}^{\frac{n}{2}} Mp_{2k} + \sum_{k=0}^{\frac{n}{2}-1} Mp_{2k+1}.$$

The even part of the above function is solved as follows. Let's start the proof using the identity (9).

$$Mp_{2k} - Mp_{2k-6} = (a+b)\, Mp_{2k-2} - abMp_{2k-4}.$$

Thence

$$\begin{aligned} Mp_6 - Mp_0 &= (a+b)\, Mp_4 - abMp_2, \\ Mp_8 - Mp_2 &= (a+b)\, Mp_6 - abMp_4, \\ Mp_{10} - Mp_4 &= (a+b)\, Mp_8 - abMp_6, \\ &\vdots \\ Mp_n - Mp_{n-6} &= (a+b)\, Mp_{n-2} - abMp_{n-4}. \end{aligned}$$

When it is summed side by side, the following equality is obtained:

$$\begin{aligned} & Mp_{n+2} + Mp_n + Mp_{n-2} - abMp_n \\ & + \begin{bmatrix} a+b-ab-1 & -1 & b-1 \\ b-1 & b-1 & -1 \\ -1 & b-a-1 & b-1 \end{bmatrix} \\ & = (a+b-ab) \sum_{k=0}^{\frac{n}{2}} Mp_{2k}. \end{aligned}$$

The odd part of the above function is also solved as follows. Using the identity (10):

$$Mp_{2k+1} - Mp_{2k-5} = (a+b)\, Mp_{2k-1} - abMp_{2k-3}.$$

Thence

$$
\begin{aligned}
Mp_7 - Mp_1 &= (a+b)\,Mp_5 - abMp_3,\\
Mp_9 - Mp_3 &= (a+b)\,Mp_7 - abMp_5,\\
Mp_{11} - Mp_5 &= (a+b)\,Mp_9 - abMp_7,\\
&\vdots\\
Mp_{n-1} - Mp_{n-7} &= (a+b)\,Mp_{n-3} - abMp_{n-5}.
\end{aligned}
$$

When it is summed side by side, the following equality is obtained:

$$
\begin{aligned}
&Mp_{n+1} + Mp_{n-1} + Mp_{n-3} - abMp_{n-1}\\
&+ \begin{bmatrix} a & a-1 & -1 \\ -1 & -1 & a-1 \\ a-1 & a^2-a-1 & -1 \end{bmatrix}\\
&= (a+b-ab)\sum_{k=0}^{\frac{n}{2}-1} Mp_{2k+1}.
\end{aligned}
$$

Using both results,

$$
\begin{aligned}
T_n = &\frac{Mp_{n+2} + Mp_{n+1} + Mp_{n-2} + Mp_{n-3} + (1-ab)\,(Mp_{n-1} + Mp_n)}{a+b-ab}\\
&+ \begin{bmatrix} \frac{2a+b-ab-1}{a+b-ab} & \frac{a-2}{a+b-ab} & \frac{b-2}{a+b-ab} \\ \frac{b-2}{a+b-ab} & \frac{b-2}{a+b-ab} & \frac{a-2}{a+b-ab} \\ \frac{a-2}{a+b-ab} & \frac{a^2+b-2a-2}{a+b-ab} & \frac{b-2}{a+b-ab} \end{bmatrix}
\end{aligned}
$$

is obtained. $\square$

4. Conclusion

Special numbers and their generalizations and applications are important in mathematics and applied sciences. In this study, a novel generalization of the Padovan sequence, called the bi-periodic Padovan sequence, was discussed, and its matrix representation was provided. The Binet-like formula, generating functions, series, partial sum formulae, and various identities were found for the bi-periodic Padovan sequences.

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References

[1] M. Alp, N. Irmak, L. Szalay, *Two-Periodic ternary recurrences and their Binetformula*, Acta Mathematica Universitatis Comenianae, 2 (2012), 227-–23.

[2] J. P. Ascano, E. N. Gueco, *The Bi-Periodic Fibonacci-Horadam Matrix*, Integers: Electronic Journal of Combinatorial Number Theory, 21 (2021), Article ID: A29, 21 pages.

[3] G. Bilgici, *Two generalizations of Lucas sequence*, Applied Mathematics and Computation, 245 (2014), 526–538.

[4] Y. Choo, *Relations between Generalized Bi-Periodic Fibonacci and Lucas Sequences*, Mathematics, 8 (9) (2020), Article ID: 1527, 10 pages.

[5] C. Cooper, *Some identities involving differences of products of generalized Fibonacci numbers*, Colloquium Mathematicum, 141 (1) (2015), 45—49.

[6] A. Coskun, N. Taskara, *A note on the bi-periodic Fibonacci and Lucas matrix sequences*, Applied Mathematics and Computation, 320 (2018), 400–406.

[7] A. Coskun, N. Taskara, *The matrix sequence in terms of bi-periodic Fibonacci numbers*, Communications Faculty of Sciences University of Ankara, Series A1: Mathematics and Statistics, 68 (2) (2016), 1939–1949.

[8] O. Deveci, E. Karaduman, *On the Padovan p-numbers*, Hacettepe Journal of Mathematics and Statistics, 46 (4) (2017), 579–592.

[9] O. Diskaya, H. Menken, *Some Identities of Gadovan Numbers*, Journal of Science and Arts, 20 (2) (2020), 317–322.

[10] O. Diskaya, H. Menken, *On the Padovan Triangle*, Journal of Contemporary Applied Mathematics, 10 (2) (2020), 77–83.

[11] M. Edson M., O. Yayenie, *A new generalization of Fibonacci sequences and extended Binet's Formula*, Integers: Electronic Journal of Combinatorial Number Theory, 9 (2009), Article ID: A48, 639–654.

[12] N. Gürses, Z. İşbilir, *On the combined Jacobsthal-Padovan generalized quaternions*, Advanced Studies: Euro-Tbilisi Mathematical Journal, 15 (2) (2022), 55–70.

[13] Z. İşbilir, N. Gürses, *Padovan and Perrin generalized quaternions*, Mathematical Methods in the Applied Sciences, 45 (18) (2022), 12060–12076.

[14] N. Irmak, L. Szalay, *On k-periodic binary recurrences*, Annales Mathematicae et Informaticae, 40 (2012), 25–35

[15] N. Irmak, M. Alp, *Some identities for generalized Fibonacci and Lucas sequences*, Hacettepe Journal of Mathematics and Statistics, 42 (4) (2013), 331–338.

[16] T. Koshy, Pell and Pell-Lucas numbers with applications, Springer, New York, 2014.

[17] T. Koshy, Fibonacci and Lucas Numbers with Applications Volume 1, John Wiley & Sons, New Jersey, 2018.

[18] T. Koshy, Fibonacci and Lucas Numbers with Applications Volume 2, John Wiley & Sons, New Jersey, 2019.

[19] J. L. Ramirez, *Bi-periodic incomplete Fibonacci sequences*, Annales Mathematicae et Informaticae, 42 (2013), 83–92.

[20] K. Sokhuma, *Matrices formula for Padovan and Perrin sequences*, Applied Mathematical Sciences, 7 (142) (2013), 7093–7096.

[21] E. Tan, S. Yilmaz, M. Sahin, *A note on bi-periodic Fibonacci and Lucas quaternions*, Chaos, Solitons & Fractals, 85 (2016), 138–142.

[22] E. Tan, H. H. Leung, *Some basic properties of the generalized bi-periodic Fibonacci and Lucas sequences*, Advances in Difference Equations, (1) (2020), 1–11.

[23] E. Tan, *Some properties of the bi-periodic Horadam sequences*, Notes on Number Theory and Discrete Mathematics, 23 (4) (2017), 56–65.

[24] E. Tan, *On bi-periodic Fibonacci and Lucas numbers by matrix method*, Ars Combinatoria, 133 (2017), 107–113,

[25] E. Tan, A. B. Ekin, *Some identities on conditional sequences by using matrix method*, Miskolc Mathematical Notes, 18 (1) (2017), 469–477.

[26] S. Uygun, E. Owusu, *A new generalization of Jacobsthal numbers (bi-periodic Jacobsthal sequences)*, Journal of Mathematical Analysis, 7 (5) (2016), 28–39.

[27] N. Yilmaz, A. Coskun, N. Taskara, *On properties of bi-periodic Fibonacci and Lucas polynomials*, In: AIP Conference Proceedings (Vol. 1863, No. 1, p. 310002), AIP Publishing LLC, 2017.

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