MATHEMATICA MORAVICA
VOL. 4 (2000), 59–65

SUBMEASURES WITH PROBABILISTIC STRUCTURES

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Abstract. In [8] the author gives some probabilistic generalizations of the submeasure concept. The purpose of this paper is to define a general form of submeasure with probabilistic structure in such way that the topological ring of sets induced is a uniform space. As particular cases the probabilistic generalizations from [8] are obtained.

1. Introduction

The study of topological set rings with the topology induced by positive submeasures is developed and systematized by L. Drewnowski in [3].

In [8], the author gives some probabilistic generalizations of the submeasure concept. The notions of probabilistic submeasures are introduced for modelling those situations in which we have only probabilistic informations about the (sub)measure of a set.

In analogy with the case of positive submeasures, the author developed a topological study of the generalized probabilistic submeasures, as well as of probabilistic structures on set rings.

The purpose of this paper is to define a general form of submeasure with probabilistic structure, in such way that the topological ring of sets induced is an uniform space. As particular case the probabilistic generalisations of submeasure notion from [8] are obtained.

The notions and notations used here follow the book [13] and the paper [8].

2. Preliminaries

Let $(S, \triangle, \cap)$ be a ring of subsets of a fixed set S with respect to the operations $\triangle$ (symmetric difference) and $\cap$ (intersection).

AMS (MOS) Subject Classification 1991. Primary: 28C99. Secondary: 54E70.
Key words and phrases: probabilistic structure, uniform space.

A mapping $\eta : S \to [0, \infty]$ is said to be a submeasure [3] if:

(i) $\eta(\emptyset) = 0$;

(ii) $\eta(A) \leq \eta(B)$ if $A, B \in S$ and $A \subset B$;

(iii) $\eta(A \cup B) \leq \eta(A) + \eta(B)$, $A, B \in S$.

For a submeasure η on S, the classes $U(\varepsilon) = \{A \in S, \eta(A) \leq \varepsilon\}$, $\varepsilon > 0$, are a normal base of neighbourhoods of $\emptyset$ for Fréchet-Nikodym topology $\tau(\eta)$ on S. This topology is semimetrizable for example by the semimetric:

$$d(A, B) = \eta(A \triangle B), \quad A, B \in S.$$

Let D_+ be the family of all submeasure distribution functions F such that $F(0) = 0$ (recall that F is nondecreasing, left continuous, and $F(\infty) = 1$). By ε_0 we denote the function of D_+ such that $\varepsilon_0(X) = 1$ for all $x > 0$.

A T-norm (in the sense of Schweizer and Sklar) is a binary operation T on $[0, 1]$ such that $([0, 1], T)$ is an order semigroup with unit. The most important t-norms are $T_1 = \text{Max}(\text{Sum} - 1, 0)$, $T_* = \text{Prod}$ and $T_\infty = \text{Min}$.

An operation (shortly O_p) is a binary operation θ on $[0, \infty)$ which is associative, commutative, nondecreasing in each place and for which $\theta(0, x) = x$ for all $x > 0$.

The $O_p - s$, $\theta_1 = \text{Sum}$ (which is the classical Sum) and $\theta_\infty = \text{Max}$ will be of principal interest in what follows.

3. Some probabilistic generalizations of the submeasure concept

In [8], the author gives the following probabilistic generalisations of the submeasure concept.

Definition 3.1. *Let S be a ring of subsets of a fixed set S and a mapping $\gamma : S \to D_+$ ($\gamma(A)$ will be denoted by γ_A) such that:*

(m_1) $A = \emptyset \iff \gamma_A(x) = \varepsilon_0(x)$, $x > 0$

(m_2) $A \subset B \Rightarrow \gamma_A(x) \leq \gamma_B(x)$, $x > 0$

(θm_3) $\gamma_{A \cup B}(\theta(x, y)) \leq T[\gamma_A(x), \gamma_B(y)] x, y > 0$, $A, B \in S$

where T is a fixed t-norm. The mapping γ is called θ-Menger submeasure. The triplet (S, γ, T) is named a θ-Menger ring.

For $\theta = \theta_1 = \text{Sum}$ the mapping γ is called Menger submeasure and for $\theta = \theta_\infty = \text{Max}$ we have the non Archimedean Menger submeasure.

Remark 3.2. If the map $\mathcal{F} : \mathcal{S} \times \mathcal{S} \to D_+$ is defined by $\mathcal{F}(A, B) = \gamma_{A \triangle B}$ then $(\mathcal{S}, \mathcal{F}, T)$ is a θ-semiMenger space under the t-norm T, and the θ-Menger semimetric on $\mathcal{S}$ is translation invariant.

Theorem 3.3. *If $(\mathcal{S},\gamma,T)$ is a ring with Menger submeasure and* $\sup_{x<1} T(x,x) = 1$, *then the family* $\{U(\varepsilon,\lambda);\ \varepsilon > 0,\ \lambda \in (0,1)\}$ $U(\varepsilon,\lambda) = \{(A,B) : \gamma_{A\triangle B}(\varepsilon) > 1-\lambda\}$ *or* $\{V(\lambda) : \lambda \in (0,1)\}$ $V(\lambda) = \{(A,B); \gamma_{A\triangle B}(\lambda) > 1-\lambda\}$ *is a base for a metrizable uniformity or* $\mathcal{S}$ *which is called* (ε,λ)-*uniformity of* $\mathcal{F}$-*uniformity.*

The proof can easily be reproduced.

As it is shown in [8] the submeasure γ can generate an uniformity on $\mathcal{S}$ in a more general situation.

Definition 3.4. *The mappings γ which verifies the axioms* (m_1), (m_2) *and* (Hm_3), *where*

$$(\forall)\varepsilon > 0, (\exists)\delta > 0 \text{ such that } 1-\gamma_A(\delta) < \delta, 1-\gamma_B(\delta) < \delta \Rightarrow$$

$$(\mathrm{Hm}_3) \qquad \Rightarrow 1-\gamma_{A\cup B}(\varepsilon) < \varepsilon, \quad A,B \in \mathcal{S}$$

is said to be a probabilistic H-submeasure.

In [8] a slight generalization of (Hm_3) formulated in terms of additive generators for t-norms is given and the notion of probabilistic f-submeasure is introduced.

Definition 3.5. *The mapping γ which verifies the axioms* (m_1), (m_2) *and* (fm_3), *where*

$$(\forall)\varepsilon > 0,\ (\exists)\delta > 0 \text{ such that } f\circ\gamma_A(\delta) < \delta,\ f\circ\gamma_B(\delta) < \delta \Rightarrow$$

$$(\mathrm{fm}_3) \qquad \Rightarrow f\circ\gamma_{A\cup B}(\varepsilon) < \varepsilon$$

and $f : [0,1] \to [0,\infty)$ *is a continuous strictly decreasing function such that* $f(1) = 0$, *is called probabilistic f-submeasure.*

Theorem 3.6. *If $(\mathcal{S},\gamma)$ is a ring with probabilistic f-submeasure, then the family* $\{V_f(\lambda);\ \lambda > 0\}$, $V_f(\lambda) = \{(A,B);\ f\circ\gamma_{A\triangle B}(\lambda) < \lambda\}$ *generates a metrizable uniformity on* $\mathcal{S}$.

The proof can be obtained without difficulty.

A natural generalization of the submeasure notion can be made according to the probabilistic generalizations of the metric notion obtained by A. N. Šerstnev using semigroup operations on D_+ namely t-functions.

A t-function (or triangular function) is a binary operation on D_+ which is associative, commutative, nondecreasing in each place and which has ε_0 as identity. An important example is $\tau = \tau_T$ defined by

$$\tau_T(F,G)(x) = \sup_{u+v=x} T(F(u),G(v))$$

where T is a left continuous t-norm.

Definition 3.7. *Let $\mathcal{S}$ be a ring of subsets of a fixed set $\mathcal{S}$. A mapping $\gamma : S \to D_+$ which verifies the axioms (m_1), (m_2) and (Sm_3) where:*

$$(Sm_3) \qquad \gamma_{A\cup B} \geq \tau(\gamma_A, \gamma_B), \quad A, B \in \mathcal{S}$$

where τ is a t-function, is called a Šerstnev submeasure under the t function τ.

Remark 3.8. If T is a left continuous T norm, and $\mathcal{F} : \mathcal{S} \times \mathcal{S} \to D_+$, $\mathcal{F}(A, B) = \gamma_{A\triangle B}$, then $(\mathcal{S}, \mathcal{F}, \tau_T)$ is a Šerstnev space if $(\mathcal{S}, \mathcal{F}, T)$ is a Menger space.

4. Submeasures with probabilistic structures

In the sequal we define a general form of submeasure with probabilistic structure in such way that the topological ring or sets induced is a uniform space.

Let $\mathcal{V}$ be a family of subsets of D_+ with the properties:

(ν_1) $a \in \mathcal{V} \Rightarrow \varepsilon_0 \in a$

(ν_2) $u \in \mathcal{V}$, $v \in \mathcal{V} \Rightarrow \exists \omega \in \mathcal{V}$, $\omega \subset u \cap v$

(ν_3) $[G \in D_+,\, G \geq F \in u \in \mathcal{V}] \Rightarrow G \in u$

that is $\mathcal{V}$ is a filter base at ε_0 on D_+ compatible with relation $\geq$.

Definition 4.1. *Let $\mathcal{S}$ be a ring of subsets of a fixed set $\mathcal{S}$. A mapping $\gamma : \mathcal{S} \to D_+$ such that:*

(m_1) $\quad A = \emptyset \iff \gamma_A(x) = \varepsilon_0(x)$, $x > 0$

(m_2) $\quad A \subset B \Rightarrow \gamma_A(x) \geq \gamma_B(x)$, $x > 0$

(PSm_3) $\quad (\forall) v \in \mathcal{V}$, $(\exists) u \in \mathcal{V}$; $(\gamma_A \in u,\, \gamma_B \in u) \Rightarrow \gamma_{A\cup B} \in v$

is called submesaure with probabilistic $\mathcal{V}$-structure and $(\mathcal{S}, \gamma, \mathcal{V})$ will be called a ring with probabilistic $\mathcal{V}$-structure.

Remark 4.2. If the map $\mathcal{F} : \mathcal{S}\times\mathcal{S} \to D_+$ is defined by $\mathcal{F}(A, B) = \gamma_{A\triangle B}$ then $(\mathcal{S}, \mathcal{F}, \mathcal{V}))$ is a probabilistic metric structure [9].

Example 4.3. (**Submeasure with probabilistic β-structure**). Let $\beta : D_+ \to [0, \infty)$ be a decreasing mapping which satisfies $\beta(\varepsilon_0) = 0$. For every $h > 0$ we define $V_h = \{F \in D_+; \beta(F) < h\}$ and let $\mathcal{V}_\beta = \{V_h\}_{h>0}$. It is obvious that $\mathcal{V}_\beta$ satisfies (ν_1), (ν_2), (ν_3).

A mapping γ which verifies (m_1), (m_2) and $(PS_\beta m_3)$, where

$(\mathrm{PS}_\beta \mathrm{m}_3)$

$$(\forall) t > 0 (\exists) h > 0; \; (\beta(\gamma_A) < h,\; \beta(\gamma_B) < h) \Rightarrow \beta(\gamma_{A\cup B}) < t, \quad A, B \in \mathcal{S}$$

will be called a submeasure with probabilistic β-structure.

a) Probabilistic H-submeasure

Let d_L be the modified Levy distance [13], defined by:

$$d_L(F,G) = \inf \Big\{ h > 0 | G(x) \le F(x+h) + h \text{ and } F(x) \le G(x+h) + h, \text{ for all } x \in \left(0, \frac{1}{h}\right) \Big\}.$$

If $\beta(F) = d_L(F, \varepsilon_0) = \inf\{h | F(h+) > 1 - h\}$ then, because $F(t) > 1 - t \iff d_L(F, \varepsilon_0) < t$ ([13] §4.3), we obtain the probabilistic H-submeasure.

b) Probabilistic f-submeasure

Let $f : [-z, 1] \to [0, \infty]$ be a fixed continuous strictly decreasing function such that $f(1) = 0$ and $f(-z) = \infty$ where $z \ge 0$ and $\mathcal{K}_f : D_+ \to [0, \infty]$ is the function defined by:

$$\mathcal{K}_f(E) = \sup\{t |\, t \le f_0 E(t)\}.$$

If $\beta = \mathcal{K}_f$ then we obtain the probabilistic f-submeasure.

Example 4.4. ($\mathcal{V}$-Šerstnev submeasure under the φ operation). Let $\mathcal{V}$ be a family of subsets of D_+ which satisfies (ν_1), (ν_2), (ν_3) and φ be a binary operation on D_+ such that $\varphi(\varepsilon_0, \varepsilon_0)$. We say that φ is $\mathcal{V}$-continuous at $(\varepsilon_0, \varepsilon_0)$ if: $(\forall) v \in \mathcal{V}$, $(\exists) u \in \mathcal{V}$, $F, G \in u \Rightarrow \varphi(F, G) \in v$.

A $\mathcal{V}$-Šerstnev submeasure under φ is a mapping $\gamma : \mathcal{S} \to D_+$ which verifies (m_1), (m_2) and $(\mathcal{V}Sm_3)$, where

$$(\mathcal{V}\text{Sm}_3) \qquad (\forall) v \in \mathcal{V},\ (\exists) u \in \mathcal{V} : \varphi(\gamma_A, \gamma_B) \in u \Rightarrow \gamma_{A \cup B} \in v, \quad A, B \in \mathcal{S}.$$

If γ is a $\mathcal{V}$-Šerstnev submeasure under φ and φ is continuous at $(\varepsilon_0, \varepsilon_0)$, then γ is a submeasure with probabilistic $\mathcal{V}$-structure.

Indeed, let $v \in \mathcal{V}$ be given and u be its correspondent from $(\mathcal{V}\text{Sm}_3)$. Since φ is continuous at $(\varepsilon_0, \varepsilon_0)$ then there exists $\omega \in \mathcal{V}$ such that $\gamma_A, \gamma_B \in \omega \Rightarrow \varphi(\gamma_A, \gamma_B) \in u$.

For this ω we have:

$$\gamma_A \in \omega,\ \gamma_B \in \omega \Rightarrow \varphi(\gamma_A, \gamma_B) \in u \Rightarrow \gamma_{A \cup B} \in v, \quad A, B \in \mathcal{S}.$$

If γ is a Šerstnev submeasure under the t-function τ then γ is a $\mathcal{V}$-Šerstnev submeasure under τ.

Since $F \ge G \Rightarrow d_L(G, \varepsilon_0) \ge d_L(F, \varepsilon_0)$ we obtain that if τ is a continuous at $(\varepsilon_0, \varepsilon_0)$ t-function (in D_+, d_L) then the Šerstnev submeasure under τ is a probabilistic H-submeasure.

5. Uniformities on ring of sets induced by submeasures with probabilistic structures

Let $(\mathcal{S}, \gamma, \nu)$ be a ring with probabilistic ν-structure. For every $v \in \nu$ we define:

$$U_v = \{(A, B) \in \mathcal{S} \times \mathcal{S};\ \gamma_{A\triangle B} \in v\} \text{ and let } \mathsf{U}_\nu^\gamma = \{U_v\}_{v\in\nu}.$$

Theorem 5.1. *U_ν^γ is a base for a uniformity on $\mathcal{S}$ (which we will call the $\gamma - \nu$-uniformity).*

Proof. It is sufficient to prove that: $(\forall)U_v \in \mathsf{U}_\nu^\gamma \Rightarrow (\exists)U_u \in \mathsf{U}_\nu^\gamma$ such that.

$$U_v \supset U_u \circ U_u = \{(A, B) \in \mathcal{S} \times \mathcal{S};\ \exists C \in \mathcal{S},\ (A, C) \in U_u,\ (C, B) \in U_u\}.$$

Indeed, let $v \in \nu$ be given and $u \in \nu$ be its corespondent from (PSm_3).

For $(A, B) \in U_u \circ U_u$, there exists $C \in \mathcal{S}$ such that

$$(A, C) \in U_u \iff \gamma_{A\triangle C} \in u \text{ and } (C, B) \in U_u \iff \gamma_{C\triangle B} \in u.$$

Then, since:

$$\gamma_{A\triangle B} = \gamma_{(A\triangle C)\triangle(C\triangle B)} \geq \gamma_{(A\triangle C)\cup(C\triangle B)} \in v$$

it follows that

$$\gamma_{A\triangle B} \in v \iff (A, B) \in U_v.$$

Corollary 5.2. *If τ is a t-function such that the mapping τ continuous at $(\varepsilon_0, \varepsilon_0)$ (in (D_+, d_L)) then for every ring with the Šerstnev submeasure under τ, the family:*

$$\mathsf{U} = \{U(\lambda)\}_{\lambda\in(0,1)},\quad U(\lambda) = \{(A, B);\ \gamma_{A\triangle B}(\lambda) > 1 - \lambda\}$$

is a uniformity base on $\mathcal{S}$.

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Received 15 Okt. 1999.