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SOME PROPERTIES OF SPACES SIMILAR TO ČECH - COMPLETE PROPERTY

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Abstract. In this paper we study some notions related to the remainder $X^* = \beta X \setminus \beta(X)$ which are similar to the Čech-complete property. A topological space X is P (wP) – *complete* if X is a Tychonoff space and remainder $X^* = \beta X \setminus \beta(X)$ is a P (wP) - set in βX. The set $A \subset X$ is an L – *set* if $A \cap cl_X(F) = \emptyset$ for each Lindelöf subset F contained in $X \setminus A$. Recall that a space X is said to be L – *complete* if X is a Tychonoff space and the remainder $X^* = \beta X \setminus \beta(X)$ is an L - set in βX.

1. Introduction

Let X be a topological space. Then:

$K(X)$ denotes the family of all nonempty compact subsets of X.

P_X denotes the set of all P-points of X.

WP_X denotes the set of all weak P-points of X.

L_X denotes the set of all L-points of X.

The closure of a subset A of a space X is denoted by $cl_X(A)$.

In this paper we assume that all spaces are Hausdorff. For notions and definitions not given here see [3], [6], [8]

Definition 1.3. *Let X be a topological space.*

(a) A point $p \in X$ is said to be a P – point if the intersection of countably many neighborhoods of p is a neighborhood of p ([8]).

(b) A point $p \in X$ is a weak$P$ – point if $p \notin cl_X(F)$ for each countable subset $F \subset X \setminus \{p\}$ (see [8]).

It is clear that every P-point is a weak P-point, it follows that $P_X \subset WP_X$. Furthermore, it can be shown that a point $p \in X$ is a P-point if and

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only if every F_σ-set that is contained in $X \setminus \{p\}$ has the closure contained in $X \setminus \{p\}$.

Definition 1.4. *Let X be a topological space.*

(a) A set $A \subset X$ is said to be a P-set if the intersection of countably many neighborhoods of A is a neighborhood of A.

(b) A set $A \subset X$ is a weak P-set($wP - set$) if $A \cap cl_X(F) = \emptyset$ for each countable set F contained in $X \setminus A$.

It is easy to see that every P-set is a weak P-set. Furthermore, the set P_X (WP_X) is a $P(wP)$-set. The converse is not necessarily true (see Example 1.4).

It is easy to see that every open set of X is a P-set.

The reader can easily prove the following lemma.

Lemma 1.5. *Let X be a topological space. The set $A \subset X$ is a P-set if and only if every F_σ-set that is contained in $X \setminus A$ has the closure contained in $X \setminus A$. If X is a compact space, then the set $A \subset X$ is a P-set if and only if every σ-compact set that is contained in $X \setminus A$ has the compact closure contained in $X \setminus A$.*

Example 1.4. Let R be the set of real numbers with the Euclidean topology.

It is known that R is second countable, separable, locally compact and σ-compact.

Every open interval (a, b) is a P-set, since for every F_σ-set $A \subset (-\infty, a] \cup [b, +\infty)$ the closure $cl_R(A) \subset (-\infty, a] \cup [b, +\infty)$. But $P_X = \emptyset$, since R is second countable.

2. $P(wP)$ complete spaces

Definition 2.1. *A topological space X is P (wP) $-$ complete if X is a Tychonoff space and the remainder $X^* = \beta X \setminus \beta(X)$ is a P (wP) - set in βX.*

Our definition of $P(wP)$ - complete spaces is an external definition; it characterizes $P(wP)$ - complete spaces by their relations to other topological spaces, viz., their compactifications. We shall now establish an internal characterization of $P(wP)$ - complete spaces. To begin, we introduce an auxiliary concept. We shall say that the space X is *hypercountably compact*(*strongly countably compact*) if every σ-compact (countable) subset of X has a compact closure.

Lemma 2.2. *A Tychonoff space X is hypercountably compact (strongly countably compact) if and only if for every compactification cX of the space X the remainder $cX \setminus c(X)$ is a $P(wP)$ - set in cX ([8]).*

Theorem 2.3. *For every Tychonoff space X the following conditions are equivalent:*

(I) For every compactification cX of the space X the remainder $cX \setminus c(X)$ is a $P(wP)$ -set in cX.

(II) The remainder $\beta X \setminus \beta(X)$ is a P (wP) -set in βX.

(III) There exists a compactification cX of the space X the remainder $cX \setminus c(X)$ is a $P(wP)$ -set in cX.

Proof. Implications $(I) \Rightarrow (II)$ and $(II) \Rightarrow (III)$ are obious, so that it suffices to prove that $(III) \Rightarrow (I)$.

$(III) \Rightarrow (I)$: $(III) \Rightarrow$ hypercountably compact (strongly countably compact) property and by Lemma 2.2, hypercountably compact(strongly countably compact) property $\Leftrightarrow (I)$.■

Remark. According to Lemma2.2, it is easy to see that the following hold:

(a $P(wP)$ - completeness is hereditary with respect to closed subspaces.

(b) The sum $\oplus\{X_s : s \in S\}$ is P (wP) - complete if and only if all spaces X_s are P (wP) - complete and the set S is finite.

(c) If there exists a continuous [perfect] mapping $f : X \longrightarrow Y$ of a wP $[P]$ - complete space X onto a Tychonoff space Y, then Y is a wP $[P]$ - complete space.

(d) Let X be the product of spaces X_a, $a \in A$.

(1) If every X_a, $a \in A$, is a P -complete space , then X is a P - complete space.

(2) The space X is wP -complete if and only if all spaces X_a are wP - complete (see [7], [8]).

Definition 2.4. *A Tychonoff space X is hemicompact if in the family of all compact subsets of X ordered by $\subset$ there exists a countable cofinal subfamily ([3], 3.4.E).*

Theorem 2.5. *For every Tychonoff space X the following conditions are equivalent:*

(I) The space X is hemicompact.

(II) For every compactification cX of the space X $\chi(cX \setminus c(X), cX) \leq \aleph_0$.

(III) There exists a compactification cX of the space X such that $\chi(cX \setminus c(X), cX) \leq \aleph_0$.

Proof. Implications $(I) \Rightarrow (II)$ and $(II) \Rightarrow (III)$ are obvious, so that it suffices to prove that $(III) \Rightarrow (I)$.

$(III) \Rightarrow (I)$: It is easy to see that $\{cX \backslash K : K \in K(X)\}$ is the family of all open neighbourhods of the subset $cX \setminus c(X)$. Since $\chi(cX \setminus c(X), cX) \leq \aleph_0$, there exists a countable subfamily $\{cX \backslash K_n : K_n \in K(X) \wedge n \in N\} \subset \{cX \backslash K : K \in K(X)\}$ such that for each $K \in K(X)$ there exists a $cX \setminus K_n, n \in N$ such that $cX \setminus K_n \subset cX \setminus K \Leftrightarrow K \subset K_n$. Hence the subfamily $\{K_n : K_n \in K(X) \wedge n \in N\}$ is cofinal in the family $K(X)$.■

Definition 2.6. *A topological space X is $\aleph_0$ – complete if X is a Tychonoff space and $\chi(\beta X \setminus \beta(X), \beta X) \leq \aleph_0$.*

The following three propositions are straitforward.

Proposition 2.7. *$\aleph_0$ - completeness is hereditary wit respect to closed subsets and with respect to G_δ - subsets.*

Proposition 2.8. *The sum $\oplus\{X_s : s \in S\}$ is $\aleph_0$ - complete if and only if all spaces X_s are $\aleph_0$ - complete and $card(S) \leq \aleph_0$.*

Proposition 2.9. *If X and Y are Tychonoff spaces and there exists a perfect mapping $f : X \longrightarrow Y$ of X onto Y, then X is $\aleph_0$ - complete if and only if Y is $\aleph_0$ - complete.*

Proposition 2.10. *Let X be the product of spaces X_i, $i \in \{1, 2 \ldots, n\}$. The space X is $\aleph_0$ - complete if and only if every X_i are $\aleph_0$ - complete.*

Proof. Case $\Rightarrow$: By Theorem 2.5, X is $\aleph_0$ - complete $\Leftrightarrow$ X is hemicompact. Let $K(X_j)$ the family of all compact subsets of $X_j, j \in \{1, 2 \ldots, n\}$. For a fixed $x \in \times\{X_i : i \in \{1, 2 \ldots, n\} \setminus \{j\}\}$ the family $A = \{K \times \{x\} : K \in K(X_j)\} \subset K(X)$ ($K(X)$ denotes the family of all nonempty compact subsets of X). Since X is hemicompact, there exists a countable cofinal subfamily $C(X) \subset K(X)$. For each $X_j \; : j \in \{1, 2 \ldots, n\}$, we have that $p_j(C) \subset p_j(A) = K(X_j)$, where p_j is the projection from X onto X_j. Furthermore, $p_j(C) \subset K(X_j)$ is countable cofinal subfamily. Hence X_j is a $\aleph_0$ - complete space.

Case $\Leftarrow$: Let $C(X_i) \subset K(X_i)$, $i \in \{1, 2, \ldots, n\}$ countable cofinal subfamily. We shall prove that $C(X) = \{\times\{C(X_i) : i \in N\}\} \subset K(X)$ is countable cofinal subfamily. For each $K \in K(X)$, we hawe that $p_i(K) \in K(X_i)$ and $p_i(K) \subset K_i \in C(X_i), i \in \{1, 2 \ldots, n\}$ (The spaces X_i is hemicompact), where p_i is the projection from X onto X_i. Furthermore, $K \subset K_1 \times K_2 \times \ldots \times K_n \in C(X) \subset K(X)$.■

3. L - complete spaces

Definition 3.1. *Let X be a topological space.*

(a) A point $p \in X$ is an L – point if $p \notin cl_X(F)$ for each Lindelöf subset $F \subset X \setminus \{p\}$.

(b) A set $A \subset X$ is an L – set if $A \cap cl_X(F) = \emptyset$ for each Lindelöf subset F contained in $X \setminus A$.

It is clear that every L-point is a weak P-point, hence $L_X \subset WP_X$. Furthermore, the set L_X is a L-set. The following example shows that no every weak P - point is an L - point.

Example 3.2. Let $[0, \omega_1]$ ($[0, \omega_0]$) be the space of ordinals less than or equal to the first uncountable ordinal (first countable ordinal) with the order topology and $[0, \omega_1] \times [0, \omega_0]$ the Cartesian product. The subspace $X_1 = [0, \omega_1] \times [0, \omega_0] \setminus \{(\omega_1, n) : n \in [0, \omega_0)\}$ of $[0, \omega_1] \times [0, \omega_0]$ is noncompact and normal in the subspace topology. Let $X_2 = X_1 \cup \{p\}$, $(p \notin X_1)$ be the one-point compactification of X_1. Then the space X_2 is compact and T_1 space. It is not Hausdorff since the point p and (ω_1, ω_0) have no disjoint neighbourhoods. The point (ω_1, ω_0) is a weak P - point but it is not a P - point.

Let $A = \{a_n \in X_2 : n \in N\}$ be any countable subset of $X_2 \setminus \{(\omega_1, \omega_0)\}$ and let $p \in A$. Then $A = \{(x_n, y_n) : x_n \in [0, \omega_1),\ y \in [0, \omega_0); n \in N\}$ where $\{x_n \in [0, \omega_1) : n \in N\} \subset [0, \omega_1)$ and $\{y_n \in [0, \omega_0) : n \in N\} \subset [0, \omega_0)$. Let a be an upper bound for the x_n; $a < \omega_1$, since ω_1 has uncountably many predecessors, while a has only countably many. Thus the set $([0, a] \times [0, \omega_0]) \cup \{p\}$ is closed and compact in X_2. Furthermore, $A \subseteq ([0, a] \times [0, \omega_0]) \cup \{p\}$ and $(\omega_1, \omega_0) \notin ([0, a] \times [0, \omega_0]) \cup \{p\}$. Then $(\omega_1, \omega_0) \notin ([0, a] \times [0, \omega_0]) \cup \{p\}$.

The point (ω_1, ω_0) is not an L - point because there exists a σ - compact (Lindelöf) subset $F = \cup\{([0, \omega_1) \times \{k\}) \cup \{p\} : k \in [0, \omega_0)\} \subset X_2$ such that $cl_{X_2}(F) = X_2$.

We need now the following simple lemma taken from 1.3.

Lemma 3.3. *Let X be a compact space. The set (point) $A \subset X (a \in X)$ is a L-set(point) if and only if every Lindelöf set that is contained in $X \setminus A$ $(X \setminus \{a\})$ has the compact closure contained in $X \setminus A$ $(X \setminus \{a\})$.*

Remark. It is clear that every L - point is a P - point, hence $L_X \subset P_X$. The following example shows that not every P - point is an L - point.

Example 3.4. Let $[0, \omega_1]$ ($[0, \omega_0]$) be the space of ordinals less than or equal to the first uncountable ordinal (first countable ordinal) with the order topology and $[0, \omega_1] \times [0, \omega_0]$ Cartesian product. The subspace $X_1 = [0, \omega_1] \times [0, \omega_0] \setminus \{(\omega_1, n) : n \in [0, \omega_0)\}$ of $[0, \omega_1] \times [0, \omega_0]$ is noncompact and normal in the subspace topology. Let $X_2 = X_1 \cup \{p\}$, $(p \notin X_1)$ be the one-point extension of X_1. We can define an topology on X_2 by declaring open base of the point p any subset of X_2 whose complement is countable. Then the space X_2 is Lindelöf and T_1 space. It is not Hausdorff (compact) since the point p and (ω_1, ω_0) have no disjoint neighbourhoods (since the subsets $([0, \omega_1) \times \{n\}) \cup \{p\}; n \in [0, \omega_0)$ are closed and noncompact subsets in X_2).

The point (ω_1, ω_0) is a P - point but not an L - point. Let $A = \{A_n \subset X_2 : n \in N\}$ be any σ - compact subset of $X_2 \setminus (\omega_1, \omega_0)$.

Case I: If $\{p\} \notin A$ then $p_1(A)$ and $p_0(A)$ are σ - compact subsets of $[0, \omega_1)$ and $[0, \omega_0]$, where p_1, p_0 are projections from $X_2 \setminus \{(\omega_1, \omega_0)\}$ onto $[0, \omega_1)$, $[0, \omega_0]$. Since $[0, \omega_1)([0, \omega_0])$ is hypercountably compact (compact) there exists a compact subsets $[0, \alpha] \subset [0, \omega_1]$ and $[0, \omega_0]$ such that $p_1(A) \subset [0, \alpha]$ and $p_0(A) \subset [0, \omega_0]$. The set $[0, \alpha] \times [0, \omega_0]$ is closed and compact in $X_2 \setminus \{(\omega_1, \omega_0)\}$. Furthermore, $A \subset [0, \alpha] \times [0, \omega_0]$ and $(\omega_1, \omega_0) \notin cl_{X_2}(A)$.

Case II: Let $\{p\} \in A$. We will now show that (ω_1, ω_0) is a P - point. According to Case I, the set $A \subset ([0, \alpha] \times [0, \omega_0]) \cup \{p\}$. Since $([0, \alpha] \times [0, \omega_0]) \cup \{p\}$ is closed and compact in $X_2 \setminus \{(\omega_1, \omega_0)\}$, the point (ω_1, ω_0) is a P - point in X_2.

The point (ω_1, ω_0) is not an L - point because there exists a Lindelöf subset $F = \cup\{([0, \omega_1) \times \{k\}) \cup \{p\} : k \in [0, \omega_0)\} \subset X_2$ such that $cl_{X_2}(F) = X_2$.

We need now the following simple lemma taken again from 1.3.

Definition 3.5. *A topological space X will be called an LC – space if each Lindelöf subspace of X has compact closure.*

Remark. The subspace $X_2 \setminus \{(\omega_1, \omega_0)\}$ in Example 3.4 is a hypercountably compact (HCC) space but it is not an LC - space.

The following is an immediate consequence of Lemma 3.3, and Definition 3.5.

Lemma 3.6. *A Tychonoff space X is an LC – space if and only if for every compactification cX of the space X the remainder $cX \setminus c(X)$ is an L - set in cX.*

Theorem 3.7. *For every Tychonoff space X the following conditions are equivalent:*

(I) For every compactification cX of the space X the remainder $cX \setminus c(X)$ is an L - set in cX.

(II) The remainder $\beta X \setminus \beta(X)$ is an L - set in βX.

(III) There exists a compactification cX of the space X the remainder $cX \setminus c(X)$ is an L - set in cX.

Proof. Implications $(I) \Rightarrow (II)$ and $(II) \Rightarrow (III)$ are obvious, so that it suffices to prove that $(III) \Rightarrow (I)$.

$(III) \Rightarrow (I) case : (III) \Rightarrow LC$ property and by Lemma 3.6, LC property $\Leftrightarrow (I)$.■ A topological space X is $L-complete$ if X is a Tychonoff space and satisfies condition (I), and hence all the conditions, in Theorem 3.7.

Proposition 3.8. *Every closed subspace of an L - complete space is L - complete.*

Proof. Since Lindelöfness is hereditary with respect to closed subsets, it immediately follows from Lemma 3.6. ■

Since compactness and Lindelöfness is hereditary with respect to closed subsets and finite unions, the following proposition is a consequence of Lemma 3.6 and Proposition 3.8.

Proposition 3.9. *The sum* $\oplus\{X_s : s \in S\}$ *is L - complete if and only if all spaces* X_s *are L - complete and the set S is finite.*

Proposition 3.10. *The Cartesian product of L - complete spaces is L - complete.*

Proof. Let $X = \times\{X_a : a \in A\}$ be the product of L - complete spaces X_a and let F be any Lindelöf subset of X. Since the projections $p_a : X \longrightarrow X_a$ from X onto X_a are continuous and open mappings, from each $a \in A$, we have that $p_a(F)$ is a Lindelöf subset of X_a. The set $cl_{X_a}(p_a(F))$ is compact in X_a. Furthermore, $F \subset Y = \times\{cl_{X_a}(p_a(F)) : a \in A\}$ and Y is a compact(closed) subspace of X. Then $cl_X(F) = cl_Y(F)$ is a compact subset of X. By Theorem 3.7 and Lemma 3.6, X is an L - complete space.■

Corollary 3.11. *The limit of an inverse sequence of L - complete spaces is L - complete.*

Proposition 3.12. *Let X be the product of spaces* X_i, $i \in \{1, 2 \ldots, n\}$. *If X is L - complete space, then every* X_i *are L - complete.*

Proof. By Theorem 3.7, and Lemma 3.6, X is L - complete $\Leftrightarrow$ X is LC - space. Let F_j be any Lindelöf subset of $X_j, j \in \{1,\ 2\ldots, n\}$. For a fixed $x \in \times\{X_i : i \in \{1,\ 2\ldots, n\} \setminus \{j\}\}$ the set $A = F \times \{x\} \subset X$ is a Lindelöf subset of X. Since X is an LC - space, $cl_X(A) \in K(X)$. For each $X_j\ : j \in \{1,\ 2, \ldots, n\}$, we have that $p_j(cl_X(A)) \in K(X_j)$, where p_j is the projection from X onto X_j. Furthermore, $F \subset p_j(cl_X(A))$ and $cl_{X_j}(F) \in K(X_j)$. According to Lemma 3.6, X_j is a L - complete space.■

Since the class of compact (Lindelöf) spaces is perfect, from the definition of L - complete spaces we obtain.

Proposition 3.13. *If X and Y are Tychonoff spaces and there exists a perfect mapping $f : X \longrightarrow Y$ of X onto Y, then X is L - complete if and only if Y is L - complete.*

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