MATHEMATICA MORAVICA
VOL. 4 (2000), 83–92

SOME CHARACTERIZATIONS OF LORENTZIAN SPHERICAL SPACELIKE CURVES WITH THE TIMELIKE AND THE NULL PRINCIPAL NORMAL

Miroslava Petrović-Torgašev and Emilija Šućurović

Abstract. In [4] the authors have characterized Lorentzian spherical spacelike curves in the 3–dimensional Minkowski space with the spacelike normal. In this paper, we shall characterize the Lorentzian spherical spacelike curves in the same space with the timelike and the null principal normal.

1. Introduction

In the Euclidean space E^3 a spherical unit speed curves and their characterizations are given in [2].

In this paper, we shall consider the Minkowski 3–space E_1^3, i.e. the space E^3 provided with the Lorentzian inner product

$$g(a,b) = -a_1b_1 + a_2b_2 + a_3b_3,$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.

Let (α) be a curve in the space E_1^3 and let α' be its tangent vector for each $s \in I \subset R$. If $g(\alpha', \alpha') > 0$ or $\alpha' = 0$ for each s then (α) is called *a spacelike curve*, if $g(\alpha', \alpha') < 0$ for each s then (α) is called *a timelike curve*, and if $g(\alpha', \alpha') = 0$ and $\alpha' \neq 0$ for each s then (α) is called *a null curve.*

The Lorentzian sphere of radius 1 in the space E_1^3 is defined by

$$S_1^2 = \{a = (a_1, a_2, a_3) \in E_1^3 : g(a,a) = 1\}.$$

In [4] the authors have characterized Lorentzian spherical spacelike curves with the spacelike principal normal. Now we shall characterize the

AMS (MOS) Subject Classification 1991. Primary: 53C50, 53C40.
Key words and phrases: Lorentzian sphere, spacelike curve, frenet formula, Minkowski space, Lorentzian inner product.

Lorentzian spherical spacelike curves with the timelike and the null principal normal.

Let $\{T, N, B\}$ be the Frenet frame of a unit speed spacelike curve $\alpha(s)$. Then $T = \alpha'(s)$, $N = \dfrac{\alpha''(s)}{\|\alpha''(s)\|}$ and B are the tangent, the principal normal and the binormal unit vectors of the curve α respectively.

If N is a timelike vector, we have that

$$g(T,T) = g(B,B) = 1, \quad g(N,N) = -1, \quad g(T,N) = g(T,B) = g(N,B) = 0,$$

and the Frenet fomulae read

$$\dot{T} = \kappa N, \quad \dot{N} = \kappa T + \tau B, \quad \dot{B} = \tau N.$$

On the other hand, if N is a null vector, we have that

$$g(T,T) = 1, \quad g(N,N) = g(B,B) = 0, \quad g(T,N) = g(T,B) = 0, \quad g(N,B) = 1,$$

and the Frenet formulae read

$$\dot{T} = \kappa N, \quad \dot{N} = \tau N, \quad \dot{B} = -\kappa T - \tau B,$$

where κ can take only two values: $\kappa = 0$ when α is a straight line or $\kappa = 1$ in all other cases. On the point of a spacelike curve $\alpha(s)$, the functions $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are called the curvature and the torsion of this curve respectively ([1]).

2. Lorentzian spherical spacelike curves with the timelike principal normal

Theorem 1. *Let $\alpha(s)$ be a unit speed spacelike curve with the timelike principal normal N, whose image lies on a Lorentzian sphere of radius r and center m in E_1^3. Then $\kappa \neq 0$ for every $s \in I \subset R$. If $\tau \neq 0$ for every $s \in I \subset R$, then*

$$\alpha - m = \frac{1}{\kappa}N - \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'B.$$

Proof. By assumption we have

$$g(\alpha - m, \alpha - m) = r^2,$$

for every $s \in I \subset R$. By differentiation in s, we find that

(1) $$g(T, \alpha - m) = 0.$$

By a new differentiation we find that

$$g(T', \alpha - m) + g(T,T) = 0,$$
$$\kappa g(N, \alpha - m) = -1.$$

Hence $\kappa \neq 0$, for all $s \in I \subset R$ and

$$g(N, \alpha - m) = -\frac{1}{\kappa}. \tag{2}$$

Next, assume that $\tau \neq 0$. Denote by

$$\alpha - m = aT + bN + cB,$$

where $a = a(s)$, $b = b(s)$, $c = c(s)$. Then by relations (1) and (2) we have

$$g(T, \alpha - m) = a = 0, \quad g(N, \alpha - m) = -b = -\frac{1}{\kappa}, \quad g(B, \alpha - m) = c. \tag{3}$$

Since $g(N, \alpha - m) = -\frac{1}{\kappa}$, by differentiation of this equation we find that

$$g(N', \alpha - m) + g(N, \alpha') = -\left(\frac{1}{\kappa}\right)',$$

$$g(\kappa T + \tau B, \alpha - m) + g(N, T) = -\left(\frac{1}{\kappa}\right)',$$

so that

$$g(B, \alpha - m) = c = -\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'.$$

and hence

$$\alpha - m = \frac{1}{\kappa}N - \frac{1}{\tau}\left(\frac{1}{\kappa}\right)' B. \square$$

Theorem 2. *Let $\alpha(s)$ be a unit speed spacelike curve, with the timelike principal normal N, with $1/\kappa \neq 0$ and $1/\tau \neq 0$ for each s. If*

$$-\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = r^2 = \text{constant},$$

where $r > 0$, then image of α lies on a Lorentzian sphere of radius r.

Proof. Consider the vector

$$m = \alpha - \frac{1}{\kappa}N + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)' B.$$

We shall prove that $m =$ constant. By differentiation in s we have that

$$\begin{aligned}
m' &= \alpha' - \left(\frac{1}{\kappa}\right)' N - \frac{1}{\kappa}N' + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' B + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)' B' = \\
&= T - \left(\frac{1}{\kappa}\right)' N - \frac{1}{\kappa}(\kappa T + \tau B) + \left(\left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)' + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)''\right) B + \left(\frac{1}{\kappa}\right)' N = \quad (4) \\
&= T - T - \frac{\tau}{\kappa}B + \left(\left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)' + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)''\right) B = \\
&= \left(-\frac{\tau}{\kappa} + \left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)' + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)''\right) B.
\end{aligned}$$

By differentiation in s of relation

$$-\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = r^2$$

we have

$$-2\Big(\frac{1}{\kappa}\Big)\Big(\frac{1}{\kappa}\Big)' + 2\Big(\frac{1}{\tau}\Big)\Big(\frac{1}{\kappa}\Big)'\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)' = 0,$$

and thus

$$-\frac{\tau}{\kappa} + \Big(\frac{1}{\tau}\Big)'\Big(\frac{1}{\kappa}\Big)' + \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'' = 0.$$

Substituting the last relation in the relation (4), we find that $m' = 0$ for each s and therefore $m =$ constant. Since

$$m = \alpha - \frac{1}{\kappa}N + \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'B,$$

we have that

$$g(\alpha - m, \alpha - m) = -\Big(\frac{1}{\kappa}\Big)^2 + \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = r^2 = \text{constant},$$

so α lies on the Lorentzian sphere with radius r whose center is m. □

Theorem 3. *If $\alpha(s)$ is a unit speed spacelike curve, with the timelike principal normal N, which satisfies $1/\kappa \neq 0$ and $1/\tau \neq 0$ for each $s \in I \subset R$, then $\alpha(s)$ lies on a Lorentzian sphere if and only if* $\frac{\tau}{\kappa} = \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'$ *and* $\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 > \Big(\frac{1}{\kappa}\Big)^2$.

Proof. Assume that $\alpha(s)$ is a curve satisfying the mentioned conditions and which lies on a Lorentzian sphere of radius r and center m. Then we have $g(\alpha - m, \alpha - m) = r^2$, for each s. By Theorem 1 we get

$$\alpha - m = \frac{1}{\kappa}N - \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'B,$$

thus

$$g(\alpha - m, \alpha - m) = -\Big(\frac{1}{\kappa}\Big)^2 + \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2.$$

It follows that

$$-\Big(\frac{1}{\kappa}\Big)^2 + \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = r^2, \tag{5}$$

so that

$$\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 > \Big(\frac{1}{\kappa}\Big)^2.$$

Differentiation of the relation (5) gives

$$-\frac{1}{\kappa}\Big(\frac{1}{\kappa}\Big)' + \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big(\Big(\frac{1}{\tau}\Big)'\Big(\frac{1}{\kappa}\Big)' + \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)''\Big) = 0,$$

and consequently

$$-\frac{\tau}{\kappa} + \Big(\frac{1}{\tau}\Big)'\Big(\frac{1}{\kappa}\Big)' + \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'' = 0.$$

Therefore

$$\frac{\tau}{\kappa} = \Big(\frac{1}{\tau}\Big)'\Big(\frac{1}{\kappa}\Big)' + \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'' = \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'.$$

Conversely, assume that $\alpha(s)$ is a unit speed spacelike curve, with the timelike principal normal N, which satisfies $1/\kappa \neq 0$ and $1/\tau \neq 0$ for each s. Further, assume that

$$\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 > \Big(\frac{1}{\kappa}\Big)^2$$

and

$$\frac{\tau}{\kappa} = \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'$$

holds.

The last equation can be easily transformed into

$$-2\Big(\frac{1}{\kappa}\Big)\Big(\frac{1}{\kappa}\Big)' + \frac{2}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)' = 0.$$

But the last expression is the differential of the equation

$$-\Big(\frac{1}{\kappa}\Big)^2 + \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = c = \text{constant},$$

so we may take that $c = r^2$ and the Theorem 2 gives that image of the curve α lies on a Lorentzian sphere of radius r. □

Theorem 4. *A unit speed spacelike curve $\alpha(s)$, with the timelike principal normal N, lies on a Lorentzian sphere if and only if $\kappa(s) > 0$ for every s and there is a differentiable function $f(s)$ such that $f\tau = (1/\kappa)', f' = \tau/\kappa$ and $|f| > 1/\kappa$.*

Proof. First assume that $\alpha(s)$ is a curve satysfying the mentioned conditions and lying on a Lorentzian sphere of radius r and center m. Then by Theorem 1 we have $\kappa \neq 0$ and

$$-\Big(\frac{1}{\kappa}\Big)^2 + \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = r^2.$$

Further, by Theorem 3 we have

$$\frac{\tau}{\kappa} = \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'.$$

Next define the differentiable function $f(s)$ by

$$f = \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'.$$

Consequently, $f' = \tau/\kappa$, and $f^2 > (1/\kappa)^2$, so $|f| > 1/\kappa$. Since $\alpha' = T$, $\alpha'' = T' = \kappa N$, we find that

$$-\kappa = g(T', N) = g\Big(T', \frac{T'}{||T'||}\Big) = -||T'||,$$

thus $\kappa = ||T'|| \geq 0$ for every s. But the Theorem 1 gives that $\kappa \neq 0$ for every s, so $\kappa > 0$ for every s.

Conversely, assume that $\alpha(s)$ is a unit speed spacelike curve with the timelike principal normal N, which satisfies $\kappa > 0$ for every s. Then $1/\kappa \neq 0$ for every s. Next, assume that there is a differentiable function $f(s)$ such that $f\tau = (1/\kappa)', f' = \tau/\kappa$ and $|f| > 1/\kappa$. Since f is differentiable, it is also continuous, so that $\tau \neq 0$ for every s. Next since

$$f = \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)',$$

we have that

$$\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)' = \frac{\tau}{\kappa},$$

so Theorem 3 implies that α lies on a Lorentzian sphere. □

Theorem 5. *A unit speed spacelike curve $\alpha(s)$, with the timelike principal normal N, lies on a Lorentzian sphere if and only if there are constants $A, B \in R$ such that*

$$\kappa\Big(A\sinh\Big(\int_0^s \tau(s)\,ds\Big) - B\cosh\Big(\int_0^s \tau(s)\,ds\Big)\Big) = 1.$$

Proof. First assume that $\alpha(s)$ is a unit speed spacelike curve with the timelike normal N, lying on the Lorentzian sphere. Then by Theorem 4 there is a differentiable function $f(s)$ such that $f\tau = (1/\kappa)', f' = \tau/\kappa$ and $|f| > 1/\kappa$. Next, define the C^2 function $\theta(s)$ and the C^1 functions $g(s)$ and $h(s)$ by

$$\theta(s) = \int_0^s \tau(s)\,ds,$$

$$g(s) = -\frac{1}{\kappa}\sinh\theta + f(s)\cosh\theta, \quad h(s) = -\frac{1}{\kappa}\cosh\theta + f(s)\sinh\theta.$$

Differentiation in s of the functions θ, g and h easily gives

$$\theta'(s) = \tau(s), \quad g'(s) = h'(s) = 0,$$

and therefore

$$g(s) = A = \text{constant}, \quad h(s) = B = \text{constant} \qquad (A, B \in R).$$

Hence we get

$$-\frac{1}{\kappa}\sinh\theta + f(s)\cosh\theta = A, \qquad -\frac{1}{\kappa}\cosh\theta + f(s)\sinh\theta = B.$$

Multiplying the first equation with $\sinh\theta$, and the second with $-\cosh\theta$ and adding we find

$$-\frac{1}{\kappa}(\sinh^2\theta - \cosh^2\theta) = A\sinh\theta - B\cosh\theta,$$

and therefore

$$\frac{1}{\kappa} = A \sinh\theta - B\cosh\theta,$$

that is

$$\kappa\Big(A\sinh\Big(\int_0^s \tau(s)\,ds\Big) - B\cosh\Big(\int_0^s \tau(s)\,ds\Big)\Big) = 1.$$

Conversely, assume that $\alpha(s)$ is a unit speed spacelike curve with the timelike normal N and there are real constants A, B such that

$$\kappa\Big(A\sinh\Big(\int_0^s \tau(s)\,ds\Big) - B\cosh\Big(\int_0^s \tau(s)\,ds\Big)\Big) = 1. \tag{6}$$

for every $s \in I \subset R$. Then obviously $\kappa \neq 0$ for every s. Differentiation in s of the relation (6) gives

$$\Big(\frac{1}{\kappa}\Big)' = \tau\Big(A\cosh\Big(\int_0^s \tau(s)\,ds\Big) - B\sinh\Big(\int_0^s \tau(s)\,ds\Big)\Big). \tag{7}$$

Next define the differentiable function $f(s)$ by

$$f(s) = A\cosh\Big(\int_0^s \tau(s)\,ds\Big) - B\sinh\Big(\int_0^s \tau(s)\,ds\Big). \tag{8}$$

Then we easily find that $|f| > 1/\kappa$. Next, the relation (7) and (8) give $(1/\kappa)' = \tau f$, that is

$$f = \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'.$$

By a new differentiation of the relation (8) and using (6) we find that

$$f' = \tau\Big(A\sinh\Big(\int_0^s \tau(s)\,ds\Big) - B\cosh\Big(\int_0^s \tau(s)\,ds\Big)\Big) = \frac{\tau}{\kappa}.$$

Therefore, the Theorem 4 gives that $\alpha(s)$ lies on a Lorentizan sphere. □

3. Lorentzian spherical spacelike curves with the null principal normal

Theorem 1. *Let $\alpha(s)$ be a unit speed spacelike curve, with the null principal normal N, in space E_1^3. Then $\alpha(s)$ lies on the Lorentzian sphere of radius r and center m if and only if $\alpha(s)$ is a planar curve and we have*

$$\alpha - m = -\frac{r^2}{2}N - B, \quad r \in R^+.$$

Proof. First assume that $\alpha(s)$ is a curve satisfying the mentioned conditions and which lies on the Lorentzian sphere of radius r and center m.

Then we have $g(\alpha - m, \alpha - m) = r^2$, for every s. By differentiation in s, we find that

$$g(T, \alpha - m) = 0. \tag{9}$$

By differentiation of the previous relation, we get

$$g(T', \alpha - m) + g(T, T) = 0,$$
$$\kappa g(N, \alpha - m) = -1.$$

and since in this case we have $\kappa = 1$ for every s, it follows that

$$g(N, \alpha - m) = -1. \tag{10}$$

By differentiation of the relation (10) and using the Frenet formulae, we find

$$\tau g(N, \alpha - m) = 0,$$

which together with the relation (10) gives $\tau = 0$, for every s. Consequently, $\alpha(s)$ is a planar curve.

Next denote by

$$\alpha - m = aT + bN + cB,$$

where $a = a(s)$, $b = b(s)$, $c = c(s)$ are arbitrary functions. Then by relations (9) and (10) we have

$$g(T, \alpha - m) = a = 0, \quad g(N, \alpha - m) = c = -1, \quad g(B, \alpha - m) = b. \tag{11}$$

By differentiation of the equation $g(B, \alpha - m) = b$, we find that

$$g(T, \alpha - m) = -b',$$

which together with the relation (9) gives $b' = 0$. Hence $b = b_0 = \text{constant} \in R$ and therefore $\alpha - m = b_0 N - B$. Since $g(\alpha - m, \alpha - m) = -2b_0 = r^2$, we find that $b_0 = -r^2/2$ and therefore

$$\alpha - m = -\frac{r^2}{2} N - B.$$

Conversely, assume that $\alpha(s)$ is a planar unit speed spacelike curve, with the null principal normal N, satisfying the equation $\alpha - m = -\frac{r^2}{2} N - B$, $r \in R^+$. Then we have

$$m = \alpha + \frac{r^2}{2} N + B$$

and by differentiation in s we find that

$$m' = \alpha' + \frac{r^2}{2} N' + B'.$$

Since in this case we have $k = 1$ and $\tau = 0$ for every s, the Frenet formulae read

$$T' = N, \quad N' = 0, \quad B' = -T.$$

Hence it follows that $m' = 0$, i.e. that the vector $m =$ constant. Therefore, we easily find that

$$g(\alpha - m, \alpha - m) = r^2,$$

so α lies on the Lorentzian sphere of radius r and center m. □

Theorem 2. *A unit speed spacelike curve $\alpha(s)$, with the null principal normal N, lies on a Lorentzian sphere if and only if there are constants $A, B \in R$ such that*

$$A \sinh\Big(\int_0^s \tau(s)\,ds\Big) - B\cosh\Big(\int_0^s \tau(s)\,ds\Big) = 1.$$

Proof. First assume that $\alpha(s)$ lies on a Lorentzian sphere. Then by Theorem 1 we have that $\tau = 0$, for every s. Next define the C^2 function $\theta(s)$ and the C^1 functions $g(s)$ and $h(s)$ by

$$\theta(s) = \int_0^s \tau(s)\,ds, \quad g(s) = -\sinh(\theta(s)), \quad h(s) = -\cosh(\theta(s)).$$

Since $\tau = 0$ for every s, we easily find that

$$\theta(s) = c = \text{constant}, \quad g(s) = -\sinh(c) = A, \quad h(s) = -\cosh(c) = B.$$

Therefore, there exist constants $A, B \in R$ such that

$$A \sinh\Big(\int_0^s \tau(s)\,ds\Big) - B\cosh\Big(\int_0^s \tau(s)\,ds\Big) = 1$$

holds.

Conversely, assume that there are constants $A, B \in R$, such that the torsion $\tau = \tau(s)$ of the curve α satisfies the equation

$$A \sinh\Big(\int_0^s \tau(s)\,ds\Big) - B\cosh\Big(\int_0^s \tau(s)\,ds\Big) = 1. \tag{12}$$

for every $s \in I \subset R$. By differentiation in s of the previous relation, we get that

$$\tau\Big(A \cosh\Big(\int_0^s \tau(s)\,ds\Big) - B\sinh\Big(\int_0^s \tau(s)\,ds\Big)\Big) = 0. \tag{13}$$

If we differentiate the last equation, we find that

$$\begin{aligned} &\tau'\Big(A \cosh\Big(\int_0^s \tau(s)\,ds\Big) - B\sinh\Big(\int_0^s \tau(s)\,ds\Big)\Big) + \\ &+\tau^2\Big(A \sinh\Big(\int_0^s \tau(s)\,ds\Big) - B\cosh\Big(\int_0^s \tau(s)\,ds\Big)\Big) = 0. \end{aligned} \tag{14}$$

Then by relations (12) and (14) we have that

$$\tau'\big(A\cosh\big(\int_0^s \tau(s)\,ds\big) - B\sinh\big(\int_0^s \tau(s)\,ds\big)\big) + \tau^2 = 0.$$

Multiplying the last equation with τ and using the relation (13), we get that $\tau^3 = 0$ and consequently $\tau = 0$, for every s.

Next, we can consider the vector

$$m = \alpha + \frac{r^2}{2}N + B,$$

where $r \in R^+$. Since $\tau = 0$ for every s, it follows that $m' = 0$ and thereby $m =$ constant. Finally, we easily obtain that

$$g(\alpha - m, \alpha - m) = r^2,$$

and consequently $\alpha(s)$ lies on the Lorentzian sphere of radius r and center m. □

4. References

[1] G. S. Birman, *Trigonometry in Lorentzian geometry*, Amer. Math. Monthly **91(9)**, (1984), 543–549.

[2] R. S. Milman, G. D. Parker, "Elements of Differential Geometry", Prentice–Hall Inc., Englwood Cliffs, New Jersey (1977).

[3] B. O'Neill, "Semi–Riemannian Geometry", Academic Press, New York (1983).

[4] U. Pekmen, S. Pasali, *Some characterizations of Lorentzian spherical space–like curves*, Mathematica Moravica **3** (1999), 31–37.

[5] I van de Woestijne, *Minimal surface of the 3–dimensional Minkowski space*, World Scientific Pub., Singapore (1990), pp.334–369.

Faculty of Science
Radoja Domanovića 12
34000 Kragujevac
Yugoslavia