MATHEMATICA MORAVICA
VOL. 4 (2000), 109–114

A NOTE ON THE POST'S COSET THEOREM

Janez Ušan and Mališa Žižović

Abstract. In this paper a proof of Post's Coset Theorem is presented. The proof uses from Theory of $n-$groups, besides the definition of $n-$groups ([[1]];1.1), the description of $n-$group as an algebra with the laws of the type $< n, n-1, n-2 >$. ([8];1.2,1.3).

1. Preliminaries

Definition 1.1. *Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. We say that (Q, A) is a Dörnte $n-$group [briefly: $n-$group] iff is an $n-$semigroup and an $n-$quasigroup as well*.*

Proposition 1.2. [8] Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. Then the following statements are equvivalent: (i) (Q, A) is an $n-$group; (ii) there are mappings $^{-1}$ and $\mathbf{e}$ respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q, \{A,^{-1}, \mathbf{e}\})$ [of the type $< n, n-1, n-2 >$]

(a) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,

(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ and

(c) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$; and

(iii) there are mappings $^{-1}$and $\mathbf{e}$ respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q, \{A,^{-1}, \mathbf{e}\})$ [of the type $< n, n-1, n-2 >$]

$(\bar{a})$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,

$(\bar{b})$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ and

$(\bar{c})$ $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$.

AMS (MOS) Subject Classification 1991. Primary: 20N15. Secondary: .

Key words and phrases: $n-$groupoids, $n-$semigroups, $n-$quasigroups,$n-$groups, $\{1, n\}-$neutral operations on $n-$groupoids, inversing operation on $n-$group.

*A notion of an $n-$group was introduced by W. Dörnte in [1] as a generalization of the notion of a group. See, also [3–5].

Remark 1.3. e *is an* $\{1,n\}-$*neutral operation of* $n-$*grupoid* (Q,A) *iff algebra* $(Q,\{A,\mathrm{e}\})$ *of type* $<n,n-2>$ *satisfies the laws* (b) *and* $(\bar{b})$ *from 1.2 [6]. Operation* $^{-1}$ *from 1.2 [(c), (*$\bar{c}$*)] is a generalization of the inversing operation in a group [7].*

Definition 1.4. Let (Q,B) be an $n-$groupoid and $n\geq 2$. Then: 1) $\overset{1}{B}\overset{def}{=}B$; and 2) for every $k\in N$ and for every $x_1^{(k+1)(n-1)+1}\in Q$

$$\overset{k+1}{B}(x_1^{(k+1)(n-1)+1})\overset{def}{=}B(\overset{k}{B}(x_1^{k(n-1)+1}),x_{k(n-1)+2}^{(k+1)(n-1)+1}).$$

Proposition 1.5. *Let* (Q,B) *be an* $n-$*semigroup,* $n\geq 2$ *and* $(i,j)\in N^2$*. Then, for every* $x_1^{(i+j)(n-1)+1}\in Q$ *and for every* $t\in\{1,\ldots,i(n-1)+1\}$ *the following equality holds*

$$\overset{i+j}{B}(x_1^{(i+j)(n-1)+1})=\overset{i}{B}(x_1^{t-1},\overset{j}{B}(x_t^{t+j(n-1)}),x_{t+j(n-1)+1}^{(i+j)(n-1)+1}).$$

2. Auxiliary proposition

Proposition 2.1 *Let* $n\geq 2$ *and let* (Q,A) *be an* $n-$*group. Also, let* $a_1^{k(n-1)},b_1^{l(n-1)},c$ *be arbitrary elements of the set* Q *such that the following equality holds*

$$\overset{k}{A}(a_1^{k(n-1)},c)=\overset{l}{A}(b_1^{l(n-1)},c)\quad [\overset{k}{A}(c,a_1^{k(n-1)})=\overset{l}{A}(c,b_1^{l(n-1)})].$$

Then for all $x\in Q$ *the following equality holds*

$$\overset{k}{A}(a_1^{k(n-1)},x)=\overset{l}{A}(b_1^{l(n-1)},x)\quad [\overset{k}{A}(x,a_1^{k(n-1)})=\overset{l}{A}(x,b_1^{l(n-1)})].$$

Sketch of the proof. $\overset{k}{A}(a_1^{k(n-1)},c)=\overset{l}{A}(b_1^{l(n-1)},c)\Rightarrow$

$$A(\overset{k}{A}(a_1^{k(n-1)},c),c_1^{n-2},(c_1^{n-2},c)^{-1})=A(\overset{l}{A}(b_1^{l(n-1)},c),c_1^{n-2},(c_1^{n-2},c)^{-1})\Rightarrow$$

$$\overset{k+1}{A}(a_1^{k(n-1)},c,c_1^{n-2},(c_1^{n-2},c)^{-1})=\overset{l+1}{A}(b_1^{l(n-1)},c,c_1^{n-2},(c_1^{n-2},c)^{-1})\Rightarrow$$

$$\overset{k}{A}(a_1^{k(n-1)},A(c,c_1^{n-2},(c_1^{n-2},c)^{-1}))=\overset{l}{A}(b_1^{l(n-1)},A(c,c_1^{n-2},(c_1^{n-2},c)^{-1}))\Rightarrow$$

$$\overset{k}{A}(a_1^{k(n-1)},\mathrm{e}(c_1^{n-2}))=\overset{l}{A}(b_1^{l(n-1)},\mathrm{e}(c_1^{n-2}))\Rightarrow$$

$$A(\overset{k}{A}(a_1^{k(n-1)},\mathrm{e}(c_1^{n-2})),c_1^{n-2},x)=A(\overset{l}{A}(b_1^{l(n-1)},\mathrm{e}(c_1^{n-2})),c_1^{n-2},x)\Rightarrow$$

$$\overset{k+1}{A}(a_1^{k(n-1)},\mathrm{e}(c_1^{n-2}),c_1^{n-2},x)=\overset{l+1}{A}(b_1^{l(n-1)},\mathrm{e}(c_1^{n-2}),c_1^{n-2},x)\Rightarrow$$

$$\overset{k}{A}(a_1^{k(n-1)},A(\mathrm{e}(c_1^{n-2}),c_1^{n-2},x))=\overset{l}{A}(b_1^{l(n-1)},A(\mathrm{e}(c_1^{n-2}),c_1^{n-2},x))\Rightarrow$$

$$\overset{k}{A}(a_1^{k(n-1)},x)=\overset{l}{A}(b_1^{l(n-1)},x)$$

[:1.2-1.5]. □

Proposition 2.2. *Let $n \geq 2$ and let (Q, A) be an $n-$group. Also, let $a_1^{k(n-1)}, b_1^{l(n-1)}, c$ be arbitrary elements of the set Q such that the following equality holds*

$$\overset{k}{A}(a_1^{k(n-1)}, c) = \overset{l}{A}(b_1^{l(n-1)}, c)$$

$$[\overset{k}{A}(c, a_1^{k(n-1)}) = \overset{l}{A}(c, b_1^{l(n-1)})].$$

Then the following equality holds

$$\overset{k}{A}(c, a_1^{k(n-1)}) = \overset{l}{A}(c, b_1^{l(n-1)}) \quad [\overset{k}{A}(a_1^{k(n-1)}, c) = \overset{l}{A}(b_1^{l(n-1)}, c)\ ].$$

Sketch of the proof. $\overset{k}{A}(a_1^{k(n-1)}, c) = \overset{l}{A}(b_1^{l(n-1)}, c) \Leftrightarrow$
$\overset{k+1}{A}(c, a_1^{k(n-1)}, c, c_1^{n-2}) = \overset{l+1}{A}(c, b_1^{l(n-1)}, c, c_1^{n-2}) \Leftrightarrow$
$A(\overset{k}{A}(c, a_1^{k(n-1)}), c, c_1^{n-2}) = A(\overset{l}{A}(c, b_1^{l(n-1)}), c, c_1^{n-2}) \Leftrightarrow$
$\overset{k}{A}(c, a_1^{k(n-1)}) = \overset{l}{A}(c, b_1^{l(n-1)})$
[:1.1 – cancellation laws, 1.5]. □

3. A proof of the Post's Coset Theorem

Theorem 3.1 (Post's Coset Theorem [2]*): *Every $n-$group has a covering group.*

Proof. Let (Q, A) be an $n-$group.

1) If $n = 2$, (Q, A) is an ordinary group and hence is its own covering group.
2) The case $n \geq 3$:

Let Γ be the set of all sequence over Q. Also, let the multiplication in Γ be defined as the juxtaposition:

$$a_1^i * b_1^j \overset{def}{=} a_1^i, b_1^j$$

for all $a_1^i, b_1^j \in \Gamma$; $i, j \in N \cup \{0\}$. Then:

1° $(\Gamma, *)$ is *a* semigroup. Moreover, $\emptyset$ [: empty sequence] is a neutral element of the semigroup $(\Gamma, *)$.

Now we define the relation θ as follows:

2° For all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Gamma$

$\boldsymbol{\alpha}\theta\boldsymbol{\beta} \overset{def}{\Leftrightarrow} (\exists\boldsymbol{\gamma} \in \Gamma)(\exists\boldsymbol{\delta} \in \Gamma)\overset{k}{A}(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}) = \overset{l}{A}(\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta})$;
$|\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}| = k(n-1)+1$, $|\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta}| = l(n-1)+1$.

By 2°,2.1 and 2.2, we conclude that the following statements holds:

*See, also [3–5].

3° Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ an arbitrary elements of the set Γ such that the statements holds: $\boldsymbol{\alpha}\theta\boldsymbol{\beta}$. Then, for each $\boldsymbol{\gamma}, \boldsymbol{\delta} \in \Gamma$ such that $|\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}| = k(n-1)+1$ and $|\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta}| = l(n-1)+1$, where $k, l \in N$, the following equality holds

$$\overset{k}{A}(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\delta}) = \overset{l}{A}(\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta}).$$

Also, the following statements hold.

4° $\theta \in Con(\Gamma, *)$.

5° Let $C(\boldsymbol{\alpha}) \cdot C(\boldsymbol{\beta}) \overset{def}{=} C(\boldsymbol{\alpha} * \boldsymbol{\beta})$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Gamma$. Then: a) $(\Gamma/\theta, \cdot)$ is a semigroup; and b) $C(\emptyset)$ is a neutral element of the semigroup $(\Gamma/\theta, \cdot)$. /See: 1° and 4°. /

6° Let $\boldsymbol{\alpha} \neq \emptyset$ and let for all $y \in Q$ the following equality holds

$$\overset{k}{A}(\boldsymbol{\alpha}, y) = y \quad [\overset{l}{A}(y, \boldsymbol{\alpha}) = y].$$

Then $\boldsymbol{\alpha} \in C(\emptyset)$.

7° For every $\boldsymbol{\alpha} \in \Gamma$ there is at least one $\boldsymbol{\beta} \in \Gamma$ /$\boldsymbol{\gamma} \in \Gamma$/ such that for all $y \in Q$ the following equality holds

$$\overset{k}{A}(\boldsymbol{\beta}, \boldsymbol{\alpha}, y) = y$$

$$/\overset{l}{A}(y, \boldsymbol{\alpha}, \boldsymbol{\gamma}) = y/.$$

8° Let $x \in Q$, $y \in Q$ and $y \in C(x)$. Then $y = x$.

9° Let $a_1^n \in Q$. Then the following equality holds

$$C(a_1) \cdot \cdots \cdot C(a_n) = C(A(a_1^n)).$$

The sketch of the proof of 4° :

a) For all $\boldsymbol{\alpha} \in \Gamma$ there is $\boldsymbol{\delta}, \boldsymbol{\varphi} \in \Gamma$ and $k \in N$ such that the following equalities hold

$$|\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi}| = k(n-1)+1 \text{ and}$$

$$\overset{k}{A}(\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi}) = \overset{k}{A}(\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi})$$

/:2°/.

b) $\overset{k}{A}(\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi}) = \overset{l}{A}(\boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\varphi}) \Rightarrow \overset{l}{A}(\boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\varphi}) = \overset{k}{A}(\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi})$

/:2°/.

c) $\overset{k}{A}(\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi}) = \overset{l}{A}(\boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\varphi}) \wedge \overset{t}{A}(\widehat{\boldsymbol{\delta}}, \boldsymbol{\beta}, \widehat{\boldsymbol{\varphi}}) = \overset{s}{A}(\widehat{\boldsymbol{\delta}}, \boldsymbol{\gamma}, \widehat{\boldsymbol{\varphi}}) \Rightarrow$

$\overset{k}{A}(\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi}) = \overset{l}{A}(\boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\varphi}) \wedge \overset{l}{A}(\boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\varphi}) = \overset{u}{A}(\boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\varphi}) \Rightarrow$

$\overset{k}{A}(\boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\varphi}) = \overset{u}{A}(\boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\varphi})$

/:2°, 3°/.

d) $\boldsymbol{\alpha}\theta\widehat{\boldsymbol{\alpha}}$, $\boldsymbol{\beta}\theta\widehat{\boldsymbol{\beta}}$, $|\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta}| = k(n-1)+1$;
$\overset{k}{A}(\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta}) = \overset{l}{A}(\boldsymbol{\gamma},\widehat{\boldsymbol{\alpha}},\boldsymbol{\beta},\boldsymbol{\delta}) = \overset{t}{A}(\boldsymbol{\gamma},\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}},\boldsymbol{\delta})$
[:2°, 3°].

The sketch of the proof of 6° :
$\overset{t}{A}(\boldsymbol{\alpha},y) = y \Rightarrow \overset{t}{A}(\boldsymbol{\alpha},\overset{k}{A}(x_1^{k(n-1)+1})) = \overset{k}{A}(x_1^{k(n-1)+1}) \Rightarrow$
$\overset{t+k}{A}(\boldsymbol{\alpha},x_1^{k(n-1)+1}) = \overset{k}{A}(\emptyset,x_1^{k(n-1)+1}) \Rightarrow \boldsymbol{\alpha}\theta\emptyset$
[:1.1 – (Q,A) is an $n-$quasigroup, 1.5, 2°].

The sketch of the proof of 7° :

Let $n\geq 3$. Then the following statements hold:

a) $\{\overset{(1)}{a}{}_1^{n-2},\ldots,\overset{(k)}{a}{}_1^{n-2},b_1^t | k\geq 0 \wedge 0\leq t < n-2 \wedge \overset{(1)}{a}{}_1^{n-2},\ldots,\overset{(k)}{a}{}_1^{n-2},b_1^t\in\Gamma\} = \Gamma$; and

b) For each $\overset{(1)}{a}{}_1^{n-2},\ldots,\overset{(k)}{a}{}_1^{n-2},b_1^t,c_{t+1}^{n-2}\in Q$ and for all $x\in Q$ the following equality holds
$\overset{k+1}{A}(\mathbf{e}(c_{t+1}^{n-2},b_1^t),c_{t+1}^{n-2},\mathbf{e}(\overset{(k)}{a}{}_1^{n-2}),\ldots,\mathbf{e}(\overset{(1)}{a}{}_1^{n-2}),\overset{(1)}{a}{}_1^{n-2},\ldots,\overset{(k)}{a}{}_1^{n-2},b_1^t,x) = x$
[:1.2–1.5]. (Remark: For $k=t=0$: $\overset{(1)}{a}{}_1^{n-2},\ldots,\overset{(k)}{a}{}_1^{n-2},b_1^t = \emptyset$.)

The sketch of the proof of 8°:

a) $y\in C(x) \Leftrightarrow y\theta x \Leftrightarrow (\exists\boldsymbol{\alpha}\in\Gamma)(\exists\boldsymbol{\beta}\in\Gamma)\overset{k}{A}(\boldsymbol{\alpha},y,\boldsymbol{\beta}) =$
$\overset{k}{A}(\boldsymbol{\alpha},x,\boldsymbol{\beta})$; $k\in N$ *[:2°]*.

b) $\overset{k}{A}(\boldsymbol{\alpha},y,\boldsymbol{\beta}) = \overset{k}{A}(\boldsymbol{\alpha},x,\boldsymbol{\beta}) \Rightarrow y = x$ *[:1.1,1.5]*.

The sketch of the proof of 9° :

a) $b = A(a_1^n) \Leftrightarrow A(b,x_1^{n-1}) = \overset{2}{A}(a_1^n,x_1^{n-1}) \Leftrightarrow b\theta a_1^n$ *[:1.1,1.5,2°]*.

b) $C(b) = C(a_1^n) = C(a_1)\cdot\cdots\cdot C(a_n)$ *[:
a), 4°, 5°]*.

c) $C(A(a_1^n)) = C(a_1)\cdot\cdots\cdot C(a_n)$ *[:a),b)]*.

By 4° – 7°, we conclude that $(\Gamma/\theta,\cdot)$ is an group.

Finally, let

$$\mathbf{A}(C(a_1),\ldots,C(a_n)) \overset{def}{=} C(a_1)\cdot\cdots\cdot C(a_n)$$

for each $C(a_1),\ldots,C(a_n)\in\{C(x)|x\in Q\}$. Also, let

$$F(a) \overset{def}{=} C(a)$$

for all $a\in Q$. Then, by 8° and 9°, we conclude that the following statements hold

1) $(\forall a_i \in Q)_1^n FA(a_1^n) = \mathbf{A}(F(a_1), \dots, F(a_n))$; and
2) F is a bijection which maps the set Q onto the set $\{C(x)|x \in Q\}$. $\square$

4. References

[1] Dörnte W.: *Untersuchengen über einen verallgemeinerten Gruppenbegriff*, Math. Z. **29** (1928), 1–19.

[2] Post E. L.: *Polyadic groups*, Trans. Amer. Math. Soc. **48** (1940), 208–350.

[3] Bruck R. H.: *A survey of binary systems,* Springer–Verlag, Berlin–Heidelberg–New York 1971.

[4] Belousov V. D.: *n–ary quasigroups,* "Stiinca", Kishinev, 1972. (In Russian).

[5] Kurosh A. G.: *General algebra* (lectures 1969–1970), "Nauka", Moskow, 1974. (In Russian).

[6] Ušan J.: *Neutral operations of n–groupoids*, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. **18–2**(1988) 117–126. (In Russian.)

[7] Ušan J.: *A comment on n–groups*, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. **24–1** (1994), 281–288.

[8] Ušan J.: *n–groups, $n \geq 2$, as variety of type $\langle n, n-1, n-2 \rangle$*, Algebra and Model Theory, Collection of papers edited by A. G. Pinus and K. N. Ponomaryov, Novosibirsk 1997, 182–208.

Institute of Mathematics
University of Novi Sad
Trg D. Obradovića 4, 21000 Novi Sad
Yugoslavia

Faculty of Tehnical Science
University of Kragujevac
Svetog Save 65, 32000 Čačak
Yugoslavia

Received 15 Dec. 1999.