A Common Fixed Point Theorem On Transversal Upper Intervally Spaces

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Dedicated to professor M. Tasković on his 60th birthday

Abstract. This paper is to present a common fixed point theorem for family of commuting mappings defined on transversal upper intervally spaces. This result extends results of M. Tasković [5].

1. Definitions and previous results

Definition of transversal intervally spaces was given by M. Tasković (see [5]).

Definition 1.1. Let \( X \) be a nonempty set. The symmetric function \( \rho : X \times X \to [a, b] \subset \mathbb{R}_0^+ \) for \( a < b \), is called an upper intervally transversal on \( X \) if there is a function \( g : [a, b] \times [a, b] \to [a, b] \) such that

\[
\rho(x, y) \leq \max \left\{ \rho(x, z), \rho(z, y), g(\rho(x, z), \rho(z, y)) \right\}
\]

for all \( x, y, z \in X \). A transversal upper intervally space is a set \( X \) together with a given upper intervally transversal on \( X \). The function \( g \) is called upper bisection function.

Definition 1.2. A mapping \( M : \mathbb{R} \to [a, b] \subset \mathbb{R}_0^+ \) for \( a < b \) is called an upper distribution function if it is nonincreasing, left-continuous with \( \inf_{x \in \mathbb{R}} M_{u,v}(x) = a \) and \( \sup_{x \in \mathbb{R}} M_{u,v}(x) = b \). We will denote by \( D \) the set of all upper distribution functions.

Definition 1.3. A transversal upper intervally T-space is a pair \( (X, \rho) \), where \( X \) is a transversal upper intervally space and where the upper intervally transversal is defined with \( \rho[u, v] = M_{u,v}(x) \) satisfying \( M_{u,v} = M_{v,u} \), \( M_{u,v}(c) = b \) for some \( c \in \mathbb{R} \), and

\[
M_{u,v}(x) = a \quad \text{for} \quad x > c \quad \text{if and only if} \quad u = v.
\]

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Examples can be found in [5].

**Definition 1.4.** (a) A sequence \((p_n)_{n \in \mathbb{N}}\) in \((X, \rho)\) converges to a point \(p \in X\) if for some \(c \in \mathbb{R}\) and for every \(\mu > c\) and every \(\sigma > 0\), there exists a natural \(M(\mu, \sigma)\), such that \(M_{p,p_n}(\mu) < a + \sigma\), whenever \(n \geq M(\mu, \sigma)\).

(b) The sequence \((p_n)_{n \in \mathbb{N}}\) will be called fundamental in \((X, \rho)\) if for some \(c \in \mathbb{R}\) and each \(\mu > c\) and every \(\sigma > 0\), there exists a natural \(M(\mu, \sigma)\), such that \(M_{p_n,p_m}(\mu) < a + \sigma\), whenever \(n, m \geq M(\mu, \sigma)\). A transversal upper intervally T-space will be called complete if each fundamental sequence in \(X\) converges to an element in \(X\).

**Definition 1.5.** A mapping \(T\) of a transversal upper intervally T-space \((X, \rho)\) into itself will be called a **interval upper contraction** iff there exists a non-decreasing function \(\varphi : [c, +\infty) \to [c, +\infty)\) for some \(c \in \mathbb{R}\) such that

\[
\text{(As)} \quad \lim_{n \to \infty} \varphi^n(t) = +\infty, \quad \text{for every} \quad t > c,
\]
satisfying the condition:

\[
\text{(Pc)} \quad M_{Tu,Tv}(x) \leq \max \left\{ M_{u,v}(\varphi(x)), M_{u,Tu}(\varphi(x)), M_{v,Tv}(\varphi(x)), M_{v,Tu}(\varphi(x)), M_{u,Tv}(\varphi(x)) \right\}
\]

for all \(u, v \in X\) and for every \(x > c\).

M. Tasković has proven the next theorem (see [5]).

**Theorem 1.1.** Let \((X, \rho)\) be a complete transversal upper intervally T-space, where the upper transverse \(\rho[u,v] = M_{u,v}(x)\) and the upper bisection function \(g : [a,b] \times [a,b] \to [a,b]\) is nondecreasing such that \(g(t,t) \leq t\) for all \(t \in [a,b]\). If \(T\) is any intervally upper contraction mapping of \(X\) into itself, then there is a unique point \(p \in X\) such that \(T\) converges to \(p\). Moreover, \(T^n q \to p\) for each \(q \in X\).

2. **Main result**

**Theorem 2.1.** Let \((X, \rho)\) be a complete transversal upper intervally T-space where the upper intervally transversal is defined with \(\rho[u,v] = M_{u,v}(x)\) and the upper intervally bisection function \(g : [a,b] \times [a,b] \to [a,b]\) is nondecreasing such that \(g(t,t) \leq t\) for every \(t \in [a,b]\). Let \((T_n)\), for \(n \in \mathbb{N}\) be a sequence of mappings from \(X\) into itself and \(S : X \to X\) be a continuous bijective function commuting with each of \(T_n\), satisfying condition \(T_n(X) \subseteq S(X)\), for all \(n \in \mathbb{N}\). Let exists a nondecreasing function \(\varphi : [c, +\infty) \to [c, +\infty)\) for some \(c \in \mathbb{R}\) such that condition \((\text{As})\) holds. If for all points \(u, v \in X\) and all mappings \(T_i\) and \(T_j\) the inequality

\[
\text{(Pcg)} \quad M_{T_iu,T_jv}(x) \leq \max \left\{ M_{S_{u,v}}^2(\varphi(x)), M_{S_{u,T_iu}}^2(\varphi(x)), M_{S_{v,T_jv}}^2(\varphi(x)), M_{S_{v,T_iu}}^2(\varphi(x))M_{S_{u,T_jv}}^2(\varphi(x)) \right\},
\]
holds for every \(x > c\), then there is a unique common fixed point \(p \in X\) for \(S\) and all of mappings \(T_n\).
Proof. Let $u_0$ be an arbitrary point from $X$. Let us define sequence $(u_n)$, for $n \in \mathbb{N}$ as follows:

\[(1) \quad u_n = S^{-1}(T_n(u_{n-1})), \quad \text{for} \quad n \in \mathbb{N}\]

We show that the sequence $v_n = S(u_n) = T_n(u_{n-1})$, for $n \in \mathbb{N}$ is fundamental in $X$.

From condition $(Pcg)$ and for all $a > c$ the next inequalities follow:

\[(2) \quad M_{Su_{n-1},Su_n}(\mu) = M_{T_{n-1}u_{n-2},T_nu_{n-1}}(\mu) \leq\]

\[\max \left\{ M_{Su_{n-2},T_{n-1}u_{n-2}}(\varphi(\mu)), M_{Su_{n-1},T_{n-1}u_{n-1}}(\varphi(\mu)), M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), \right.\]

\[\left. M_{Su_{n-2},T_{n-1}u_{n-1}}(\varphi(\mu))M_{Su_{n-1},T_{n-1}u_{n-2}}(\varphi(\mu)), \quad M_{Su_{n-2},Su_{n-1}}(\varphi(\mu))M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)) \right\} =\]

\[\max \left\{ M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M_{Su_{n-1},Su_n}(\varphi(\mu)), \right.\]

\[\left. M_{Su_{n-2},Su_{n-1}}(\varphi(\mu))M_{Su_{n-1},Su_{n-1}}(\varphi(\mu)), \quad M_{Su_{n-2},Su_{n-1}}(\varphi(\mu))M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)) \right\}.\]

Since the space is a transversal upper intervally space then for every $x \geq c$ the following inequalities hold:

\[(*) \quad M_{u,v}(x) \leq \max \left\{ M_{u,w}(x), M_{w,v}(x), g(M_{u,w}(x), M_{w,v}(x)) \right\} \leq \max \left\{ M_{u,w}(x), M_{w,v}(x) \right\},\]

because $g(u, v) \leq g(\max\{u, v\}, \max\{u, v\}) \leq \max\{u, v\}$. From previous follows that

\[(3) \quad M_{Su_{n-2},Su_n}(\varphi(\mu)) \leq \max\{M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M_{Su_{n-1},Su_n}(\varphi(\mu))\} \].

Then, from inequality (3) and the fact that values of upper distribution functions are in interval $[a, b]$ next inequalities follow:

\[(4) \quad M_{Su_{n-2},Su_n}(\varphi(\mu))M_{Su_{n-1},Su_{n-1}}(\varphi(\mu)) = M_{Su_{n-2},Su_n}(\varphi(\mu)) \leq \]

\[\max \left\{ M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M_{Su_{n-1},Su_n}(\varphi(\mu)) \right\} \leq \]

\[\max \left\{ M_{Su_{n-2},Su_{n-1}}^2(\varphi(\mu)), M_{Su_{n-1},Su_n}^2(\varphi(\mu)) \right\}.\]

\[(5) \quad M_{Su_{n-2},Su_n}(\varphi(\mu))M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)) \leq \]

\[\max \left\{ M_{Su_{n-2},Su_{n-1}}^2(\varphi(\mu)), M_{Su_{n-1},Su_n}(\varphi(\mu))M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)) \right\}.\]
From the fact that $\max\{u^2, v^2, uv\} = \max\{u^2, v^2\}$, for all $u, v \in [a, b]$, inequalities (2), (4) and (5) imply:

$$M_{S_{n-1}, S_n}^2(\mu) \leq \max \left\{ M_{S_{n-2}, S_{n-1}}^2(\phi(\mu)), M_{S_{n-1}, S_n}^2(\phi(\mu)) \right\}.$$  \hspace{1cm} (6)

From last follows:

$$M_{S_{n-1}, S_n}(\mu) \leq \max \left\{ M_{S_{n-2}, S_{n-1}}(\phi(\mu)), M_{S_{n-1}, S_n}(\phi(\mu)) \right\}.$$ \hspace{1cm} (7)

Since $\phi$ is a nondecreasing function and $\phi(\mu) > c$, $\phi(\mu) > \mu$ for every $\mu > c$ it follows by induction that for every $k \in \mathbb{N}$ the following inequality holds:

$$M_{S_{n-1}, S_n}(\mu) \leq \max \left\{ M_{S_{n-2}, S_{n-1}}(\phi(\mu)), M_{S_{n-1}, S_n}(\phi^k(\mu)) \right\},$$ and when $k \to +\infty$ we get that for every $n \in \mathbb{N}$:

$$M_{S_{n-1}, S_n}(\mu) \leq M_{S_{n-2}, S_{n-1}}(\phi(\mu)).$$ \hspace{1cm} (9)

By induction we can prove the inequality (10) for the sequence $\{v_n\}$.

$$M_{v_{n-1}, v_n}(\mu) \leq M_{v_0, v_1}(\phi^{n-1}(\mu)).$$ \hspace{1cm} (10)

From (\ast), and last inequality, for $m > n$ and arbitrary $\mu > c$, follows:

$$M_{v_n, v_m}(\mu) \leq \max \left\{ M_{v_n, v_{n+1}}(\mu), \ldots, M_{v_{m-1}, v_m}(\mu) \right\} \leq \max \left\{ M_{v_0, v_1}(\phi^n(\mu)), \ldots, M_{v_0, v_1}(\phi^{m-1}(\mu)) \right\} = M_{v_0, v_1}(\phi^n(\mu)).$$

From (As) we conclude that exists a natural $\mathcal{M}(\mu, \sigma)$ such that $M_{v_0, v_1}(\phi^{\mathcal{M}(\mu, \sigma)}(\mu)) < a + \sigma$. We can take that $n, m \geq \mathcal{M}(\mu, \sigma)$ and we conclude that $v_n$ is a fundamental sequence in $(X, \rho)$. Since the space is complete, then there is a point $p \in X$ such that $v_n \to p$.

We shall prove that $p$ is a common fixed point for $S$ and $T_n$. Since $S$ commutates with each of $T_n$, then from (1) and the fact that $T_nS_{n-1} = ST_{n}u_{n-1} = SS_{n}$ follows:

$$M_{SS_{n-1}, T kp}(\mu) = M_{ST_{n}u_{n-1}, T kp}(\mu) = M_{T_{n}u_{n-1}, T kp}(\mu) \leq \max \left\{ M_{SS_{n-1}, Sp}(\phi(\mu)), M_{SS_{n-1}, T kp}(\phi(\mu)), M_{S_{n-1}, T kp}(\phi(\mu)), M_{SS_{n-1}, T kp}(\phi(\mu))M_{S_{n-1}, T kp}(\phi(\mu)) \right\} = \max \left\{ M_{SS_{n-1}, Sp}(\phi(\mu)), M_{SS_{n-1}, SS_{n}}(\phi(\mu)), M_{S_{n-1}, T kp}(\phi(\mu))M_{SS_{n-1}, T kp}(\phi(\mu)) \right\}.$$
From continuity of \( S \) and because \( Su_n \to p \) when \( n \to +\infty \), we get that for every \( k \in \mathbb{N} \) follows:

\[
M^2_{Sp,T_kp}(\mu) \leq \max \left\{ M^2_{Sp,Sp}(\varphi(\mu)), M^2_{Sp,Sp}(\varphi(\mu)), M^2_{Sp,T_kp}(\varphi(\mu)), M_{Sp,T_kp}(\varphi(\mu)) M_{Sp,Sp}(\varphi(\mu)) \right\} = M^2_{Sp,T_kp}(\varphi(\mu)).
\]

Because all of the functions in last inequality are nonincreasing we conclude that for each \( m \in \mathbb{N} \) the inequality \( M_{Sp,T_kp}(\mu) \leq M_{Sp,T_kp}(\varphi^m(\mu)) \) holds. When \( m \to +\infty \), for every \( \mu > c \), we obtain \( M_{Sp,T_kp}(\mu) = a \). From this, for every \( k \in \mathbb{N} \) we obtain \((**)\) \( S(p) = T_k(p) \). In following text we shall show that \( p \) is a common fixed point for all of mappings \( T_n \).

From inequality:

\[
M^2_{Su_n,T_kp}(\mu) = M^2_{T_i u_{n-1},T_kp}(\mu) \leq \max \left\{ M^2_{Su_{n-1},Sp}(\varphi(\mu)), M^2_{Su_{n-1},Su_n}(\varphi(\mu)), M^2_{Sp,T_kp}(\varphi(\mu)), M_{Su_{n-1},T_kp}(\varphi(\mu)) M_{Su_{n-1},Su_n}(\varphi(\mu)) \right\},
\]

when \( n \to +\infty \), because \((***)\) holds, we conclude that:

\[
M^2_{p,T_kp}(\mu) \leq \max \left\{ M^2_{p,T_kp}(\varphi(\mu)), M^2_{p,p}(\varphi(\mu)), M^2_{T_kp,T_kp}(\varphi(\mu)), M_{p,T_kp}(\varphi(\mu)) F_{T_kp,p}(\varphi(\mu)), M_{p,T_kp}(\varphi(\mu)) M_{p,p}(\varphi(\mu)) \right\},
\]

From last, we obtain that for each \( \mu > c \) holds the following:

\[
M_{p,T_kp}(\mu) \leq M_{p,T_kp}(\varphi(\mu)).
\]

Next, we obtain that for every \( m \in \mathbb{N} \) follows \( M_{p,T_kp}(\mu) \leq M_{p,T_kp}(\varphi^m(\mu)) \), and when \( m \to +\infty \), we conclude that for every \( \mu > c \) the fact \( M_{p,T_kp}(\mu) = a \) holds, and it implies that for each \( k \in \mathbb{N} \) we get \( p = T_kp = Sp \).

Let us prove uniqueness of common fixed point \( p \). Suppose that there is another common fixed point \( q \neq p \). From

\[
M^2_{p,q}(\mu) = M^2_{T_i p,T_j q}(\mu) \leq \max \left\{ M^2_{Sp,Sq}(\varphi(\mu)), M^2_{Sp,p}(\varphi(\mu)), M^2_{Sq,q}(\varphi(\mu)), M_{Sp,q}(\varphi(\mu)) M_{Sq,p}(\varphi(\mu)) M_{Sp,p}(\varphi(\mu)) \right\} = M^2_{p,q}(\varphi(\mu)).
\]

follows that for every \( \mu > c \) holds that \( M_{p,q}(\mu) \leq M_{p,q}(\varphi(\mu)) \), and so, for every \( m \in \mathbb{N} \), we obtain that \( M_{p,q}(\mu) \leq M_{p,q}(\varphi^m(\mu)) \), and when \( m \to +\infty \), we conclude that for every \( \mu > c \) holds the fact \( M_{p,q}(\mu) = a \). From conditions for distribution functions we get that \( p = q \). This completes the proof. \( \square \)

**Comment.** It is easy to prove that from condition \((Pcg)\) for \( T = T_i = T_j \) and \( S = I \), where \( I \) is an identical mapping, follows:
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\[ M^2_{T_u,T_v}(x) \leq \max \left\{ M^2_{u,v}(\varphi(x)), M^2_{u,T_u}(\varphi(x)), M^2_{v,T_v}(\varphi(x)) \right\} \leq \max \left\{ M^2_{u,v}(\varphi(x)), M^2_{u,T_u}(\varphi(x)), M^2_{v,T_v}(\varphi(x)) \right\}, \]

and we can conclude that mappings satisfying condition (Pcg) are intervally upper contractions.

From these conclusions follows that Theorem 2 is an extension of Theorem 1.

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References

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