ON THE \(p\)-REDUCED ENERGY OF A GRAPH

MIRJANA LAZIĆ

Abstract. Let \(G\) be a simple connected graph of order \(n\) and let \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\) be the spectrum of \(G\). Then the sum \(S^l_k(G) = |\lambda_{k+1}| + |\lambda_{k+2}| + \cdots + |\lambda_{n-l}|\) is called \((k, l)\)-reduced energy of \(G\), where \(k, l\) are two fixed nonnegative integers [2]. In this work, we make a generalization of the \((k, l)\)-reduced energy, as follows: for any fixed \(p \in \mathbb{N}\), the sum \(S^l_k(G, p) = |\lambda_{k+1}|^p + |\lambda_{k+2}|^p + \cdots + |\lambda_{n-l}|^p\) is called the \(p\)-th \((k, l)\)-reduced energy of the graph \(G\). We also here introduce definitions of some other kinds of the \(p\)-reduced energies and we prove some properties of them.

1. Introduction

In this paper we consider only simple connected graphs. The vertex set of a graph \(G\) is denoted by \(V(G)\), and its order by \(|G|\). The spectrum of such a graph is the set \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\) of eigenvalues of its 0-1 adjacency matrix [1].

Let \(N_0\) be the set of all nonnegative integers and \(l \in N_0\) be a fixed number. For any graph \(G\) with \(|G| = n > l\) the sum of eigenvalues \(|\lambda_1| + |\lambda_2| + \cdots + |\lambda_{n-l}|\) is denoted by \(S^l(G)\) and is called \(l\)-positive reduced energy of \(G\) [2]. In this work we shall define the sum \(S^l(G, p) = |\lambda_{k+1}|^p + |\lambda_{k+2}|^p + \cdots + |\lambda_{n-l}|^p\) which is called the \(p\)-th \(l\)-positive reduced energy of the graph \(G\). It contains at least the largest eigenvalue \(\lambda_1(G)\) of \(G\). We note that \(|\lambda_1| \geq 1\), hence \(S^l(G, p) \geq 1\) for any graph \(G\). For any real \(a \geq 1\) and any \(l \in N_0\) and \(p \in \mathbb{N}\), we can consider the class of graphs \(E^l(p, a) = \{G : S^l(G, p) \leq a\}\).

Now, we prove an important property of the general class \(E^l(p, a)\).

Theorem 1. For every constant \(a \geq 1\) and for any fixed \(l \in N_0\) and \(p \in \mathbb{N}\), the class of connected graphs \(E^l(p, a) = \{G : |\lambda_1|^p + \cdots + |\lambda_{n-l}|^p \leq a\}\) is finite.

Proof. Let \(a\) be any real number \((a \geq 1)\) and \(l\) be nonnegative integer. Let \(G\) be a graph of order \(n > l\) from the class \(E^l(p, a)\). Then

\[(1) \quad S^l(G, p) = |\lambda_1|^p + \cdots + |\lambda_{n-l}|^p \leq a,\]
which provides that
\begin{equation}
\sum_{|\lambda_i| \geq 1} |\lambda_i|^p \leq \sum_{i=1}^{n-l} |\lambda_i|^p + \sum_{i=n-l+1}^{n} |\lambda_i|^p \leq a + \sum_{i=n-l+1}^{n} |\lambda_i|^p = a + l \cdot |\lambda_1|^p \leq a(l + 1)
\end{equation}

Relations (1) and (2) now give
\begin{equation}
2(n - 1) \leq 2m = \sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{|\lambda_i| < 1} |\lambda_i| + \sum_{|\lambda_i| \geq 1} |\lambda_i|^p \leq (n - 1) \cdot 1 + a(l + 1)
\end{equation}

where \(m\) is the number of edges of the graph \(G\). Using (3) we find that \(n \leq a(l + 1) + 1\), which completes the proof. \(\square\)

Next, let \(k, l \in N_0\) and \(p \in N\) be any fixed numbers. For any graph \(G\) with \(|G| = n > k + l\), the sum of eigenvalues \(|\lambda_{k+1}| + |\lambda_{k+2}| + \cdots + |\lambda_{n-l}|\) is denoted by \(S_k^l\) and is called \((k, l)\)-reduced energy of \(G\). The sum of \(p\)-th degrees of eigenvalues \(|\lambda_{k+1}|^p + |\lambda_{k+2}|^p + \cdots + |\lambda_{n-l}|^p\) is denoted by \(S_k^l(G, p)\) and is called the \(p\)-th \((k, l)\)-reduced energy of the graph \(G\). For any real \(a > 0\), for any \(k, l \in N_0\), and for \(p \in N\), we can consider the class of graphs
\[E_k^l(p, a) = \{G : S_k^l(G, p) \leq a\}.\]

We note that the \(p\)-th \((0, k)\)-reduced energy of \(G\) is \(p\)-th \(k\)-positive reduced energy of \(G\). We can always assume that \(k, l \in N_0\) and \(p \in N\).

Since the complete bipartite graph \(K^n_m\) belongs to the class \(E_k^l(p, a)\) for any \(m, n \in N\), the class \(E_k^l(p, a)\) is always infinite. In what follows, we will prove an important property of this kind of energy on the set so-called canonical graphs.

We say that two vertices \(x, y \in V(G)\) are equivalent in \(G\) and write \(x \sim y\) if \(x\) is non-adjacent to \(y\), and \(x\) and \(y\) have exactly the same neighbors in \(G\). Relation \(\sim\) is an equivalence relation on the vertex set \(V(G)\). The corresponding quotient graph is denoted by \(\tilde{G}\), and is called the canonical graph of \(G\).

Consequently, for any real \(a > 0\) and \(k, l, p \in N\), we can consider the class of the corresponding canonical graphs
\[\tilde{E}_k^l(p, a) = \{G : G \in E_k^l(p, a)\} \text{ is a canonical graph}\].

If \(k = l\), then \(S_k^k(p, a), E_k^k(p, a), \tilde{E}_k^k(p, a)\) are simply denoted by \(S_k(p, a), E_k(p, a), \tilde{E}_k(p, a)\), respectively.

We now prove an important property of the class \(\tilde{E}_k^l(p, a)\) \((a > 0, k, p \in N)\). It is based on two theorems proved in [3].

**Theorem 2.** For every constant \(a > 0\) and any positive integers \(k, p \in N\), the class of connected graphs \(\tilde{E}_k^l(p, a)\) is finite.

**Proof.** On the contrary, we shall suppose that for some \(a > 0\) (\(a\) is a positive integer) and \(k, p \in N\) the set \(\tilde{E}_k^l(p, a)\) is infinite. By Theorem proved in [3], for
any real number \( A > 0 \), there exists a graph \( G \in \tilde{E}_k(p,a) \), which has \( q > A \) nonzero eigenvalues. This graph will satisfy the relation
\[
(4) \quad |\lambda_{k+1}|^p + |\lambda_{k+2}|^p + \cdots + |\lambda_{n-k}|^p \leq a.
\]
Suppose that \( \lambda_r > \lambda_{r+1} = \cdots = \lambda_{r+s} = 0 > \lambda_{r+s+1} \), where \( s = n-q \) is the multiplicity of zero of this graph. The characteristic polynomial of \( G \) is then
\[
P_n(\lambda) = \lambda^s(\lambda^q + a_1\lambda^{q-1} + \cdots + a_q),
\]
where \( |a_q| = \lambda_1 \cdots \lambda_r \cdot |\lambda_{r+s+1}| \cdots |\lambda_n| \).

By Theorem also proved in [3], we shall suppose that \( s = 0 \) and thus \( n = q \). Also, we can assume that \( n \) is so large that we have \( \sqrt{n} \geq a + 6k \).

It is clear that \( |\lambda_i| \leq n-1 \) for \( i \in \{1,2,\ldots,k\} \cup \{n-k+1,n-k+2,\ldots,n\} \), and \( |\lambda_i| \leq \sqrt{n} \) for \( i = k+1,k+2,\ldots,n-k \).

We can assign with \( t \) the total number of eigenvalues \( \lambda_i \), with \( |\lambda_i| \leq 1/\sqrt{n} \) \( (i = k+1,k+2,\ldots,n-k) \). It is easy to see that \( t > a + 4k \). On the contrary, we would have that there exist at least \( n-(2k+t) \) eigenvalues \( \lambda_i (k+1 \leq i \leq n-k) \) with \( |\lambda_i| > 1/\sqrt{n} \). By relation (4) we have
\[
(5) \quad a \geq \sum_{i=k+1}^{n-k} |\lambda_i|^p > \frac{n-(2k+t)}{\sqrt{n}} > \frac{\sqrt{n} - a + 6k}{\sqrt{n}}.
\]

From relation \( \sqrt{n} \geq a + 6k \) and relation (5) we have \( a > \sqrt{n} - 1 \) what is a contradiction.

Denote with \( t_0 \) the total number of all eigenvalues \( \lambda_i (i = k+1,k+2,\ldots,n-k) \) with \( |\lambda_i| > 1 \). By relation (4) we have
\[
(6) \quad a \geq |\lambda_{k+1}|^p + \cdots + |\lambda_{n-k}|^p \geq \sum_{|\lambda_i|^p > t_0} |\lambda_i|^p \geq t_0,
\]
which provides that \( t_0 \leq a \).

We now have
\[
|a_n| = (|\lambda_1| \cdots |\lambda_k|)(|\lambda_{k+1}| \cdots |\lambda_{n-k}|)(|\lambda_{n-k+1}| \cdots |\lambda_n|) \leq (n-1)^{2k} \frac{\sqrt{n} \cdots \sqrt{n}}{t_0} \frac{1}{\sqrt{n}} \cdots \frac{1}{\sqrt{n}} \frac{1}{n-(t+t_0+2k)}
\]
which is a contradiction \( (|a_n| \in \mathbb{N} \quad a_n \neq 0) \). Consequently, the set \( \tilde{E}_k(p,a) \) is finite for every \( A > 0 \) and every \( k,p \in \mathbb{N} \).

**Corollary 1.** For every constant \( A > 0 \) and any positive integers \( k,l,p \in \mathbb{N} \), the class of connected graphs \( \tilde{E}_k^l(p,a) \) is finite.

**Proof.** Without loss of generality, we can assume that \( k \geq l \). Let \( G \) be any graph from the class \( \tilde{E}_k^l(p,a) \). Since
\[
a \geq \sum_{i=k+1}^{n-l} |\lambda_i|^p = \sum_{i=k+1}^{n-k} |\lambda_i|^p + \sum_{i=n-k+1}^{n-l} |\lambda_i|^p \geq \sum_{i=k+1}^{n-k} |\lambda_i|^p
\]
we have $G \in \tilde{E}_k(p,a)$, thus $\tilde{E}_k^l(p,a) \subseteq \tilde{E}_k(p,a)$. Since the class $\tilde{E}_k(p,a)$ is finite for every $a > 0$ and every $k, p \in \mathbb{N}$, we get the statement. \hfill \Box

References

